4. [12 points] Consider the continuous, piecewise-defined function $r(x)$ given as follows:

$$r(x) = \begin{cases} -x^2 - 2x & \text{if } x < 0 \\ xe^{-x} & \text{if } x \geq 0 \end{cases}.$$

For each part below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

It may be helpful to note that $e \approx 2.72$ and/or that $\frac{1}{e} \approx 0.37$.

a. [4 points] Find all critical points of $r(x)$. Show all your work.

Solution:

$$r'(x) = \begin{cases} -2x - 2 & \text{if } x < 0 \\ -xe^{-x} + e^{-x} = e^{-x}(1-x) & \text{if } x > 0 \end{cases}$$

To find critical points, we need to find where $r'(x) = 0$ and where $r(x)$ is not differentiable. Using the first piece of $r'(x)$, we have $-2x - 2 = 0$ when $x = -1$. (This is less than 0 so is indeed in the domain of the first piece of the function.) From the second piece, we have $e^{-x}(x-1) = 0$ when $x = 1$. (This is greater than 0 so is indeed in the domain of the second piece of the function.)

Finally, we need to check differentiability at $x = 0$. We are told r is continuous (though we can also quickly check that both pieces of $r(x)$ evaluate to 0 when $x = 0$). However, note that plugging $x = 0$ into the two pieces of $r'(x)$ gives -2 for the first piece and e^{-1} for the second piece. Hence there is a sharp corner at $x = 0$, which is therefore also a critical point of $r(x)$. so $r'(x) = 0$ at $x = -1$ and 1. Also, $r'(x)$ DNE at 0 because $-2 \neq 1$.

Answer: Critical point(s) at $x =$ $-1, 0, 1$

b. [4 points] Find the x-coordinates of all global minimum(s) and global maximum(s) of $r(x)$ **on the interval $[-2, 1]$**. If there are none of a particular type, write NONE.

Solution: Since $r(x)$ is continuous on the closed interval $[-2, 1]$, the Extreme Value Theorem guarantees a global max and a global min. To find these, we create a table of values of $r(x)$ for all critical points in and endpoints of the domain $[-2, 1]$.

x	$r(x)$
-2	0
-1	1
0	0
1	$e^{-1} = \frac{1}{e} \approx 0.37$

We see that the maximum value of 1 is attained at $x = -1$ and the minimum value of 0 is attained at both $x = -2$ and $x = 0$.

Answer: Global max(es) at $x =$ -1

Answer: Global min(s) at $x =$ -2 **and** 0

c. [4 points] Find the x-coordinates of all global minimum(s) and global maximum(s) of $r(x)$ **on the interval $(-\infty, \infty)$**. If there are none of a particular type, write NONE.

Solution: We create a table showing the value of $r(x)$ at all critical points along with the end behavior of $r(x)$ on the domain $(-\infty, \infty)$. Since $r(x)$ decreases without bound as $x \to -\infty$, there is no global minimum of $r(x)$. However, $r(x)$ attains a global maximum value of 1 at $x = -1$.

x	$r(x)$
$\lim_{x\to-\infty}$	$-\infty$ (DNE)
-1	1
0	0
1	$e^{-1} = \frac{1}{e} \approx 0.37$
$\lim_{x\to\infty}$	0

Answer: Global max(es) at $x =$ -1

Answer: Global min(s) at $x =$ NONE