6. [8 points] Aziz is trying to finalize his donut frosting recipe, which uses only butter and sugar. He makes frosting in 3-pound batches, so if he uses x pounds of butter and y pounds of sugar, he needs

$$x + y = 3.$$

He believes that the number of donuts D he will sell each year, in thousands, will depend on his frosting recipe, and in particular, that D can be modeled by the function

$$D = \frac{x^3}{3} + 2xy + 3y - 8.$$

What values of x and y will maximize the number of donuts he sells? Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the number of donuts sold.

Solution: The constraint is $3 = x + y$, so $y = 3 - x$. Plugging in gives

$$D(x) = \frac{x^3}{3} + 2x(3-x) + 3(3-x) - 8 = \frac{x^3}{3} - 2x^2 + 3x + 1.$$

Then taking the derivative and setting it equal to zero, we have

$$\begin{aligned} D'(x) &= x^2 - 4x + 3 = (x-3)(x-1) \\ 0 &= (x-3)(x-1). \end{aligned}$$

D' always exists, so the only critical points are $x = 1, 3$, and domain is $[0, 3]$, since we need $x \geq 0$ and $y \geq 0$. Plugging in,

x	$D(x)$
0	1
1	$\frac{1}{3} + 2$
3	1

Thus the maximum occurs when $x = 1$ and $y = 2$.

Alternate Solution: The constraint is $3 = x + y$, so $x = 3 - y$. Plugging in gives

$$D(y) = \frac{(3-y)^3}{3} + 2(3-y)y + 3y - 8.$$

Then

$$\begin{aligned} D'(y) &= -(3-y)^2 + 2(3-y) - 2y + 3 \\ 0 &= -(y^2 - 6y + 9) - 4y + 9 \\ 0 &= -y^2 + 2y \\ 0 &= -y(y-2) \end{aligned}$$

So critical points are $y = 0, 2$, and domain is $[0, 3]$ since we need $y \geq 0$ and $x \geq 0$. Plugging in,

x	$D(x)$
0	1
2	$\frac{1}{3} + 2$
3	1

Thus the maximum occurs when $x = 1$ and $y = 2$.