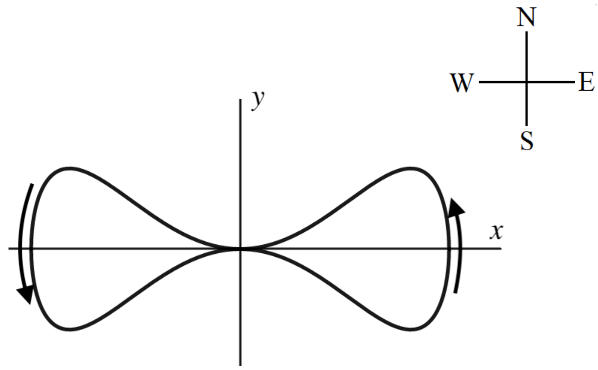
9. [7 points]

You are on a hiking trip, following the path modeled by the curve $\mathcal{B}$ defined by the equation

$$y^2 = x^4(1 - x^2).$$

Note that

$$\frac{dy}{dx} = \frac{x^3(2 - 3x^2)}{y}.$$



The graph of $\mathcal{B}$ is shown to the right. You begin your hike at $(0, 0)$, then:

- travel East and around the loop on the right as shown by the arrow, returning to $(0, 0)$, then
- travel West and around the loop on the left as shown by the arrow, returning to $(0, 0)$.

a. [5 points] Using calculus, find the coordinates of all the other points (x, y) on your path (that is, other than $(0, 0)$), where you travel directly East or directly West. Show your work. *Note that you can use the graph to determine how many points you are looking for.*

Solution: We look for where the numerator of $\frac{dy}{dx}$ is 0, i.e. $x^3(2 - 3x^2) = 0$. We ignore the solution $(0, 0)$, so we need $2 - 3x^2 = 0$ or $x = \pm\sqrt{\frac{2}{3}}$. Then

$$y^2 = \left(\pm\sqrt{\frac{2}{3}}\right)^4 \left(1 - \left(\pm\sqrt{\frac{2}{3}}\right)^2\right)$$

$$y^2 = \frac{4}{9}\left(1 - \frac{2}{3}\right) = \frac{4}{9} \cdot \frac{1}{3} = \frac{4}{27}$$

so $y = \pm\frac{2}{3\sqrt{3}}$. We can see which direction we are traveling at these points from the graph.

Answer: travel East at $\left(\sqrt{\frac{2}{3}}, -\frac{2}{3\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, -\frac{2}{3\sqrt{3}}\right)$

Answer: travel West at $\left(\sqrt{\frac{2}{3}}, \frac{2}{3\sqrt{3}}\right), \left(-\sqrt{\frac{2}{3}}, \frac{2}{3\sqrt{3}}\right)$

b. [2 points] Using calculus, find the coordinates of all the points (x, y) on your path where you travel directly North or directly South. Note that, as shown by the graph, $(0, 0)$ is not one of these points. Show your work.

Solution: We look for where the denominator of $\frac{dy}{dx}$ is 0, i.e. $y = 0$. Then $0 = x^4(1 - x^2)$ so either $x = 0$ or $x = \pm 1$. We ignore $(0, 0)$, so the points are $(1, 0)$ and $(-1, 0)$.

Answer: travel North at $(1, 0)$

Answer: travel South at $(-1, 0)$