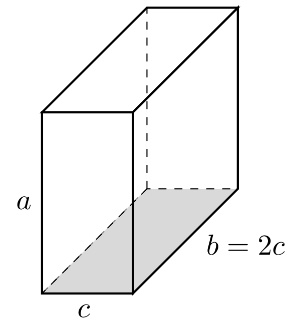
2. [9 points] A company is designing a new line of suitcases with height a, length b, and width c, all in inches. The dimensions of the new suitcases need to satisfy the constraints

$$b = 2c \qquad \text{and} \qquad a + b + c = 45.$$

What are the dimensions of such a suitcase with the largest possible volume, and what is this maximum volume?

Use calculus to find and justify your answer, and be sure to show enough evidence that the values you find do in fact maximize the volume of the suitcase.



Solution: We want to maximize volume, V, where

$$V = abc = a(2c)c = 2ac^2.$$

In order to maximize V, we must first express it in terms of a single variable. We can do this by solving for a in terms of c using the constraint equation $a + b + c = 45$:

$$a = 45 - b - c = 45 - 2c - c = 45 - 3c.$$

Substituting $a = 45 - 3c$ into our expression for V, we get

$$V = 2ac^2 = 2(45 - 3c)c^2 = 90c^2 - 6c^3.$$

Now that we have expressed V in terms of the single variable c, we can solve the problem by maximizing V over an appropriate domain. Since both c and V should be positive, we have $0 < c < 15$, so we must **maximize $V(c) = 90c^2 - 6c^3$ over the domain $(0, 15)$.**

Differentiating, we get

$$\frac{dV}{dc} = 180c - 18c^2 = 18c(10 - c).$$

Setting $\frac{dV}{dc}$ equal to zero and solving gives us the critical points $c = 0$ and $c = 10$. Since $90c^2 - 6c^3$ is differentiable everywhere, there are no other critical points.

Now we test V at $c = 0, 10, 15$. Since $V(0) = V(15) = 0$ and $V(10) = 2(45 - 3 \cdot 10) \cdot 10^2 = 3000$, we see that the maximum of $V(c)$ on $(0, 15)$ occurs at $c = 10$, with a value of $V(10) = 3000$. Solving for a and b using the constraint equations gives $b = 20$ and $a = 15$.

Answer: The volume is maximized when $a =$ 15 in., $b =$ 20 in., and $c =$ 10 in.,

and the maximum volume is 3000 cubic inches.