

6. [6 points] Let \mathcal{C} be the curve implicitly defined by the equation $xy = y^2 + 2x$. Note that

$$\frac{dy}{dx} = \frac{2-y}{x-2y}.$$

- a. [3 points] Find the coordinates of all points on the curve \mathcal{C} where the tangent line to \mathcal{C} is horizontal. If no such points exist, write DNE and show work to justify your answer.

Solution: The curve \mathcal{C} has a horizontal tangent line at points where $\frac{dy}{dx} = 0$. From the given equation $\frac{dy}{dx} = \frac{2-y}{x-2y}$, we see that this happens when $y = 2$ and $x \neq 4$. Substituting $y = 2$ into the equation that defines \mathcal{C} , we get

$$2x = 2^2 + 2x,$$

which has no solutions. It follows that there are *no* points on the curve \mathcal{C} where \mathcal{C} has a horizontal tangent line.

Answer: _____ **DNE** _____

- b. [3 points] Find the coordinates of all points on the curve \mathcal{C} where the tangent line to \mathcal{C} is vertical. If no such points exist, write DNE and show work to justify your answer.

Solution: The curve \mathcal{C} has a vertical tangent line at points where $\frac{dx}{dy} = 0$. Since $\frac{dx}{dy} = \frac{x-2y}{2-y}$, this happens when $x = 2y$ and $y \neq 2$. Substituting $x = 2y$ into the equation that defines \mathcal{C} gives

$$2y^2 = y^2 + 4y, \quad \text{which is equivalent to } y(y-4) = 0.$$

This has solutions $y = 0$ and $y = 4$. Plugging these values for y into $xy = y^2 + 2x$, we see that $x = 0$ when $y = 0$, and $x = 8$ when $y = 4$.

Answer: _____ **(0, 0) and (8, 4)** _____

7. [5 points] The equation $\sin(x^3) + x^2y = 1 + y^2$ defines y implicitly as a function of x .

Find a formula for $\frac{dy}{dx}$ in terms of x and y . Show every step of your work.

Solution: Implicitly differentiating this equation with respect to x gives

$$\cos(x^3) \cdot 3x^2 + 2xy + x^2 \frac{dy}{dx} = 2y \frac{dy}{dx}.$$

Collecting the terms involving $\frac{dy}{dx}$ on one side, we get

$$\cos(x^3) \cdot 3x^2 + 2xy = 2y \frac{dy}{dx} - x^2 \frac{dy}{dx} = (2y - x^2) \frac{dy}{dx}.$$

Finally, after dividing both sides by $2y - x^2$ we get $\frac{dy}{dx} = \frac{3x^2 \cos x^3 + 2xy}{2y - x^2}$.

Answer: $\frac{dy}{dx} =$ _____ $\frac{3x^2 \cos x^3 + 2xy}{2y - x^2}$ _____