6. [6 points] Let $\mathcal{C}$ be the curve implicitly defined by the equation $xy = y^2 + 2x$. Note that

$$\frac{dy}{dx} = \frac{2-y}{x-2y}.$$

a. [3 points] Find the coordinates of all points on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is horizontal. If no such points exist, write DNE and show work to justify your answer.

Solution: The curve $\mathcal{C}$ has a horizontal tangent line at points where $\frac{dy}{dx} = 0$. From the given equation $\frac{dy}{dx} = \frac{2-y}{x-2y}$, we see that this happens when $y = 2$ and $x \neq 4$. Substituting $y = 2$ into the equation that defines $\mathcal{C}$, we get

$$2x = 2^2 + 2x,$$

which has no solutions. It follows that there are *no* points on the curve $\mathcal{C}$ where $\mathcal{C}$ has a horizontal tangent line.

Answer: **DNE**

b. [3 points] Find the coordinates of all points on the curve $\mathcal{C}$ where the tangent line to $\mathcal{C}$ is vertical. If no such points exist, write DNE and show work to justify your answer.

Solution: The curve $\mathcal{C}$ has a vertical tangent line at points where $\frac{dx}{dy} = 0$. Since $\frac{dx}{dy} = \frac{x-2y}{2-y}$, this happens when $x = 2y$ and $y \neq 2$. Substituting $x = 2y$ into the equation that defines $\mathcal{C}$ gives

$$2y^2 = y^2 + 4y, \qquad \text{which is equivalent to} \quad y(y-4) = 0.$$

This has solutions $y = 0$ and $y = 4$. Plugging these values for y into $xy = y^2 + 2x$, we see that $x = 0$ when $y = 0$, and $x = 8$ when $y = 4$.

Answer: $(0, 0)$ **and** $(8, 4)$

7. [5 points] The equation $\sin(x^3) + x^2 y = 1 + y^2$ defines y implicitly as a function of x.

Find a formula for $\frac{dy}{dx}$ in terms of x and y. Show every step of your work.

Solution: Implicitly differentiating this equation with respect to x gives

$$\cos(x^3) \cdot 3x^2 + 2xy + x^2 \frac{dy}{dx} = 2y \frac{dy}{dx}.$$

Collecting the terms involving $\frac{dy}{dx}$ on one side, we get

$$\cos(x^3) \cdot 3x^2 + 2xy = 2y\frac{dy}{dx} - x^2\frac{dy}{dx} = (2y - x^2)\frac{dy}{dx}.$$

Finally, after dividing both sides by $2y - x^2$ we get $\dfrac{dy}{dx} = \dfrac{3x^2 \cos x^3 + 2xy}{2y - x^2}$.

Answer: $\frac{dy}{dx} =$ $\dfrac{dy}{dx} = \dfrac{3x^2 \cos x^3 + 2xy}{2y - x^2}$