9. [10 points] The continuous function $w(x)$ is defined piecewise for all real numbers by the rule

$$w(x) = \begin{cases} -x^2+3x+1 & x < -1 \\ 3x^{1/3} & -1 \le x \le 1 \\ -x^2+3x+1 & x > 1. \end{cases}$$

a. [5 points] Find the x-coordinates of all critical points of $w(x)$. If there are none, write NONE. Show your work.

Solution: We must find all x for which $w'(x)$ is either zero or does not exist. We have

$$w'(x) = \begin{cases} 3-2x & x < 1; \\ x^{-2/3} & -1 < x < 1; \\ 3-2x & x > 1; \\ \text{DNE} & x = -1; \\ 1 & x = 1. \end{cases}$$

Note that $w'(-1)$ DNE since $3-2(-1) \neq (-1)^{-2/3}$, while $w'(1) = 1$ since $3-2(1) = 1 = (1)^{-2/3}$. Now, $3-2x = 0$ when $x = \frac{3}{2}$, and $x^{-2/3}$ is never zero but is undefined when $x = 0$. Thus the critical points of $w(x)$ are:

$$x = -1, \; x = 0, \; \text{and } x = 3/2.$$

Answer: Critical point(s) at $x =$ **-1, 0, and $3/2$**

b. [3 points] Let $L(x)$ be the linear approximation of the function $w(x)$ at the point $x = \frac{1}{2}$. Find a formula for $L(x)$. Your answer should not include the letter w, but you do not need to simplify.

Solution: The linear approximation of $w(x)$ at $x = \frac{1}{2}$ is

$$\begin{aligned} L(x) &= w\left(\frac{1}{2}\right) + w'\left(\frac{1}{2}\right)\left(x - \frac{1}{2}\right) \\ &= 3\left(\frac{1}{2}\right)^{1/3} + \left(\frac{1}{2}\right)^{-2/3}\left(x - \frac{1}{2}\right) = \frac{3}{\sqrt[3]{2}} + \sqrt[3]{4}\left(x - \frac{1}{2}\right). \end{aligned}$$

Answer: $L(x) = 3\left(\frac{1}{2}\right)^{1/3} + \left(\frac{1}{2}\right)^{-2/3}\left(x - \frac{1}{2}\right)$

c. [2 points] Does $L(x)$ give an overestimate or underestimate for $w(x)$ near $x = \frac{1}{2}$? Circle your answer below, and show work to justify your answer.

UNDERESTIMATE $\boxed{\text{OVERESTIMATE}}$

Solution: $L(x)$ gives an overestimate for $w(x)$ near $x = \frac{1}{2}$ if $w(x)$ is concave down near $\frac{1}{2}$, and an underestimate if $w(x)$ is concave up near $\frac{1}{2}$. We can determine the concavity of $w(x)$ near $\frac{1}{2}$ by looking at the sign of $w''(x)$. Near $x = \frac{1}{2}$, we have

$$w'(x) = x^{-2/3}, \quad \text{so} \quad w''(x) = -\left(\frac{2}{3}\right)x^{-4/3} \quad \text{and} \quad w''\left(\frac{1}{2}\right) = -\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)^{-4/3} < 0.$$

Thus $w(x)$ is **concave down** near $x = \frac{1}{2}$, so $L(x)$ gives an **overestimate**.

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