6. (12 points) The electric field (in Newtons/Coulomb) outside of a charged sphere of charge q (in Coulombs) is given by the formula

$$E(r) = \frac{kq}{r^2}$$

where k is a positive constant and r is the distance measured in meters from the center of the sphere to the point from which one is measuring.

(a) Find a formula for the local linearization of $E(r)$ near $r = 2$ meters. [Your answer will contain k and q.]

Denote the local linearization of $E(r)$ by $L(r)$. Then note that $E'(r) = -\frac{2kq}{r^3}$. Thus we have that $E'(2) = -\frac{kq}{4}$ and $E(2) = \frac{kq}{4}$, which is all the information we need to determine the local linearization. So $L(r) = \frac{kq}{4} - \frac{kq}{4}(r-2)$.

(b) Use your result from part (a) to approximate $E(2.1)$. [Again, your answer will contain k and q.]

The local linearization provides an approximation to the function near the point 2 in this case. Therefore we have $E(2.1) \approx L(2.1) = \frac{kq}{4} - \frac{kq}{4}(2.1-2) = \frac{9kq}{40}$.

(c) Assuming $q > 0$, do you expect your estimate in part (b) to be an over- or underestimate of the actual value of $E(2.1)$? Use calculus to justify your answer. Explain.

We use the concavity of the function $E(r)$ to answer this question. Note that $E''(2) = \frac{6kq}{2^4} > 0$ since k, q are both positive constants. Thus the function is concave up at the point 2. Looking at a graph of a concave up function it is easy to see that the tangent line is an underapproximation in this case. Therefore our estimate should be an underapproximation to the actual value $E(2.1)$.

7. (10 points) While exploring an exotic spring break location, you discover a colony of geese who lay golden eggs. You bring 20 geese back with you. Suppose each goose can lay 294 golden eggs per year. You decide maybe 20 geese isn't enough, so you consider getting some more of these magical creatures. However, for each extra goose you bring home there are less resources for all the geese. Therefore, for each new goose the amount of eggs produced will decrease by 7 eggs per goose per year. How many more geese should you bring back if you want to maximize the number of golden eggs per year laid? Show your work.

Let x be how many more geese you bring back. So the total number of geese you have is $20 + x$. The number of eggs a single goose can lay in a year is given by $294 - 7x$. Therefore, the total number of eggs produced is

$$A(x) = (20 + x)(294 - 7x).$$

This is the function we would like to maximize. So we take its derivative:

$$\begin{aligned} A'(x) &= 1(294 - 7x) + (20 + x)(-7) \\ &= 154 - 14x \end{aligned}$$

Setting this equal to 0 to find the critical points we obtain $0 = 154 - 14x$. Thus the only critical point is at $x = 11$. To check that this is actually a maximum, we take the second derivative of $A(x)$ which is $A''(x) = -14$, so this is a maximum. Therefore we should bring back 11 more geese to maximize the number of golden eggs produced each year.