5. (14 points) A family of functions is given by $r(x) = \dfrac{a}{x}e^{bx}$ for a, b, and $x > 0$.

(a) For what values of a and b does the graph of r have a local minimum at the point $(4, 5)$? Show your work and **all supporting evidence** that your function satisfies the given properties.

We begin by computing $r'(x)$:
By the product rule, $r'(x) = \frac{ab}{x}e^{bx} + \frac{-a}{x^2}e^{bx}$
$= \frac{a}{x}e^{bx}(b - \frac{1}{x})$.
Now, the local minimum of r will occur when the derivative is equal to zero. So we set $r'(4)$ equal to zero: $0 = \frac{a}{4}e^{4b}(b - \frac{1}{4})$. Since $\frac{a}{4}e^{4b} > 0$ for all values of $a > 0$ and $b > 0$, we can just set $(b - \frac{1}{4}) = 0$. Doing so yields $b = \frac{1}{4}$.
Now, we must find a. We are given that the point $(4, 5)$ is on the graph of r. So we plug in $x = 4$ and $b = \frac{1}{4}$ into the equation for r and set it equal to 5:
$5 = \frac{a}{4}e^{(\frac{1}{4})4}$.
Solve for a to get that $a = \frac{20}{e}$.
We must now check to make sure that these values of a and b do indeed make the graph of r have a local minimum at the point $(4, 5)$. To check this, we compute the second derivative of r, plug in our values for a and b, and then plug in $x = 4$. This will tell us the concavity of the graph of r at $(4, 5)$, which will in turn, tell us whether we have a local maximum or a local minimum.
$r''(x) = \frac{\frac{20}{e}}{x}e^{\frac{1}{4}x}(\frac{1}{4} - \frac{1}{x})(\frac{1}{4} - \frac{1}{x}) + \frac{\frac{20}{e}}{x}e^{\frac{1}{4}x}(\frac{1}{x^2})$
$= \frac{20}{ex}e^{\frac{1}{4}x}((\frac{1}{4} - \frac{1}{x})^2 + \frac{1}{x^2})$.
Since $\frac{20}{ex}e^{\frac{1}{4}x} > 0$ for $x > 0$, it suffices to look at the sign of $((\frac{1}{4} - x)^2 + \frac{1}{x^2})$. Plugging in $x = 4$ here gives $(\frac{1}{4} - \frac{1}{4})^2 + \frac{1}{4^2} > 0$. Thus we have that the graph of r is concave up at $x = 4$ and our values of a and b do indeed make r have a local minimum at $x = 4$.

(b) Write an explicit formula for $r(x)$. Circle your answer.

From (a), we know that $r(x) = \frac{20}{ex}e^{\frac{x}{4}} = \frac{20}{x}e^{(\frac{x}{4}-1)}$.

(c) Is the graph of r concave up or down for $x > 0$? Explain using arguments based on calculus–not only from a graph.

From (a), we know that $r''(x) = \frac{a}{x}e^{bx}((b - \frac{1}{x})^2 + \frac{1}{x^2})$. Since $a > 0$, $b > 0$, and $x > 0$, we know that $\frac{a}{x}e^{bx} > 0$. So it suffices to check the sign of $((b - \frac{1}{x})^2 + \frac{1}{x^2}) + \frac{1}{x^2}$. $(b - \frac{1}{x})^2 > 0$ for all $x > 0$ and $\frac{1}{x^2} > 0$ for all $x > 0$. So putting these pieces together, we see that $r''(x) > 0$ for all $x > 0$ and thus the graph of r is concave up for $x > 0$.