7. (10 points) The University has made an agreement with the Student Government Association to sell more student season football tickets. The tickets will cost \$150 each for the first 20,000 tickets. After 20,000 have sold, students will sign up for tickets. For each additional student (over 20,000) that signs up, the season price will be reduced by \$0.01 (yes, one cent) per student. A maximum of 35,000 total student tickets will be set aside. Students may sign up for tickets until August 20th. Effective on August 21st, the additional students may pick up their tickets at the reduced rate that has been determined by the number of students who had signed up by August 20th.

(a) What total number of student sales maximizes the university's revenue from student season football tickets? Show your work. [For full credit, you must show the function(s) you use for this problem. Just plugging numbers into a table will not suffice. In addition, show evidence of the use of calculus to find your answer–not merely a graph. State clearly what any variables in your function(s) represent.]

We know that the university's revenue will be given by the price per ticket times the number of tickets sold. Notice that for the first $20,000$ tickets, the price per ticket is fixed. Let t be the number of tickets sold, and let R be the university's revenue from student season football tickets. So for $0 \le t \le 20,000$, $R(t) = 150t$. Let's now consider what happens after $20,000$ students purchase a ticket. We need to calculate the price per ticket. For each additional student (over $20,000$), the price will be reduced by \$0.01. The number of additional students is $t - 20,000$, so the total price reduction (just for the additional students) is given by $0.01(t-20,000)$. Therefore, the price per ticket (for the additional students) is $150 - 0.01(t - 20,000)$. So the university's revenue for the additional tickets is $(150 - 0.01(t - 20,000))(t - 20,000)$. Their total revenue is the revenue from the first $20,000$ tickets sold plus the revenue from the additional tickets sold: $(150 - 0.01(t - 20,000))(t - 20,000) + 150(20,000)$. Thus,

$$R(t) = \begin{cases} 150t & 0 \le t \le 20,000 \\ (150 - 0.01(t-20,000))(t-20,000) + 150(20,000) & 20,000 < t \le 35,000 \end{cases}.$$

To find the maximum, we will analyze the two parts of the function $R(t)$ separately. Notice that the first part ($0 \le t \le 20,000$) is a linear function with positive slope, so its maximum occurs at $t = 20,000$. The maximum value is $20,000(150) = 3$ million dollars. To calculate the maximum of the other portion of the function, we take the derivative of $R(t) = (150 - 0.01(t - 20,000))(t - 20000) + (150)(20,000) = -0.01t^2 + 550t - 4,000,000$:
$R'(t) = -0.02t + 550$. Setting this equal to zero yields $t = 27,500$. We can see that this is a maximum since $R''(t) = -0.02 < 0$ for all values of $t \ge 20,000$. Notice that $R(27,500) = 3,562,500$ dollars. Since this is greater than $R(20,000)$, the maximum occurs at $t = 27,500$.

(b) What is the maximum revenue from student season ticket sales (based on this problem)?

From the solution above to (a), this is \$3,562,500.

(c) How many additional (i.e., over 20,000) season tickets would be optimal financially from the students' point of view? (Consider here only the students over the initial 20,000.) Explain.

From the work done in (a), the cost of an additional student's ticket is given by the function $c(t) = 150 - 0.01(t - 20,000) = -0.01t + 350$. We want to minimize this function. Notice that we are dealing with a linear function, so the minimum will occur when t is greatest, ie. when $t = 35,000$. We can see that when $t = 35,000$, $c(35,000) = 0$ and thus the additional students would get free tickets. From the students' point of view $15,000$ additional season tickets would be optimal.