3. (18 points)

(a) (2 pts) If \( f(x) = ax^4 - x^3 + d \) \((a \neq 0)\) and \( f \) has a global maximum, what must be the sign of \( a \)? Explain.

The sign of \( a \) must be negative for \( f \) to have a global maximum. If \( a \) were positive, then \( f \to \pm \infty \) as \( x \to \pm \infty \), eliminating the possibility of a global maximum.

(b) (4 pts) Determine all critical points of \( f \).

- \( f'(x) = 4ax^3 - 3x^2 = x^2(4ax - 3) \),
- \( f \) is a polynomial, so the only critical points are those \( x \) for which \( f'(x) = 0 \),
- \( f'(x) = 0 \Rightarrow x = 0 \) or \( x = 3/(4a) \).

Hence, the critical points are \( x = 0 \) and \( x = 3/(4a) \), \( a < 0 \).

(c) (4 pts) For what value of \( x \) does the maximum occur? Show your work.

Since

\[
f''(x) = 12ax^2 - 6x = x(12ax - 6),
\]

- \( f''(3/(4a)) = 9/(4a) < 0 \) (as \( a < 0 \)), so there is a local maximum at \( x = 3/(4a) \);
- \( f''(0) = 0 \), so the second derivative test does not work for this critical point. Note that for \( x > 3/(4a) \), \( f' \) is negative, so \( x = 0 \) is neither a max or a min. Since the end behavior of \( f \) is toward \( -\infty \) as \( x \to \pm \infty \), and \( x = 3/(4a) \) is the only local max, it is the global max.

(d) (4 pts) For what value(s) of \( x \) (if any) does \( f \) have inflection points?

Since

\[
f''(x) = 12ax^2 - 6x = x(12ax - 6) = 0 \quad \text{for} \quad x = 0 \quad \text{or} \quad x = 1/(2a),
\]

these are the two possible inflection points. We test the sign of \( f'' \) to the left and the right of each of these \( x \) values. Since \( a < 0 \), we have \( 1/(2a) < 0 \), also,

\[
f'' < 0 \quad \text{for} \quad x < 1/(2a),
\]
\[
f'' > 0 \quad \text{for} \quad 1/(2a) < x < 0,
\]

and

\[
f'' < 0 \quad \text{for} \quad x > 0.
\]

This means \( x = 0 \) and \( x = 1/(2a) \) are both inflection points, as the sign of \( f'' \) changes around those points.

(e) (4 pts) If \( f(0) = 4 \) and \( f \) has a critical point at \( x = -1/4 \), determine a formula for \( f(x) \).

- \( f(0) = 4 \Rightarrow d = 4 \);
- Since the only critical points are \( x = 0 \) and \( x = 3/(4a) \), we must have:
  \[3/(4a) = -1/4 \Rightarrow a = -3.\]

So this means,

\[
f(x) = -3x^4 - x^3 + 4.
\]