3. (18 points)

(a) (2 pts) If $f(x) = ax^4 - x^3 + d$ $(a \neq 0)$ and f has a global maximum, what must be the sign of a? Explain.

The sign of a must be **negative** for f to have a global maximum. If a were positive, then $f \to \pm\infty$ as $x \to \pm\infty$, eliminating the possibility of a global maximum.

(b) (4 pts) Determine all critical points of f.

- $f'(x) = 4ax^3 - 3x^2 = x^2(4ax - 3)$,
- f is a polynomial, so the only critical points are those x for which $f'(x) = 0$,
- $f'(x) = 0 \Rightarrow x = 0$ or $x = 3/(4a)$.

Hence, the critical points are $x = 0$ and $x = 3/(4a)$, $a < 0$.

(c) (4 pts) For what value of x does the maximum occur? Show your work.

Since

$$f''(x) = 12ax^2 - 6x = x(12ax - 6),$$

- $f''(3/(4a)) = 9/(4a) < 0$ (as $a < 0$), so there is a local maximum at $x = 3/(4a)$;
- $f''(0) = 0$, so the second derivative test does not work for this critical point. Note that for $x > 3/(4a)$, f' is negative, so $x = 0$ is neither a max or a min. Since the end behavior of f is toward $-\infty$ as $x \to \pm\infty$, and $x = 3/(4a)$ is the only local max, it is the global max.

(d) (4 pts) For what value(s) of x (if any) does f have inflection points?

Since

$$f''(x) = 12ax^2 - 6x = x(12ax - 6) = 0 \quad \text{for } x = 0 \text{ or } x = 1/(2a),$$

these are the two possible inflection points. We test the sign of f'' to the left and the right of each of these x values. Since $a < 0$, we have $1/(2a) < 0$, also,

$$f'' < 0 \quad \text{for } x < 1/(2a),$$

$$f'' > 0 \quad \text{for } 1/(2a) < x < 0,$$

and

$$f'' < 0 \quad \text{for } x > 0.$$

This means $x = 0$ and $x = 1/(2a)$ are both inflection points, as the sign of f'' changes around those points.

(e) (4 pts) If $f(0) = 4$ and f has a critical point at $x = -\dfrac{1}{4}$, determine a formula for $f(x)$.

- $f(0) = 4 \Rightarrow d = 4$;
- Since the only critical points are $x = 0$ and $x = 3/(4a)$, we must have: $3/(4a) = -1/4$ or $a = -3$.

So this means,

$$f(x) = -3x^4 - x^3 + 4.$$