4. Suppose that $x$ and $y$ satisfy the relation given by the curve

$$x^4 + y^3 = 2 + \frac{7}{2}xy$$

(a) (5 points) Find $\frac{dy}{dx}$.

Differentiating both sides of the above equation with respect to $x$, we get

$$4x^3 + 3y^2\frac{dy}{dx} = \frac{7}{2}\left(y + x\frac{dy}{dx}\right)$$

Solving for $\frac{dy}{dx}$, we have

$$\frac{dy}{dx} = \frac{\frac{7}{2}y - 4x^3}{3y^2 - \frac{7}{2}x.}$$

(b) (3 points) Under what condition(s) (if any) on $x$ and $y$ is the tangent line to the curve horizontal?

The tangent line to the curve is horizontal where $\frac{dy}{dx} = 0$ and the resulting point is on the curve.

The numerator of the derivative is zero when $\frac{7}{2}y - 4x^3 = 0$. Solving for $y$ in terms of $x$ we find that $y = \frac{8}{7}x^3$.

Note that we don't want the numerator to be zero at the same time, so there is a horizontal tangent if $y = \frac{8}{7}x^3$ and $x \neq \frac{6y^2}{7}$.

(c) (2 points) Consider the points $(1, 2)$ and $(3, 4)$. One of these points lies on the curve, and one does not. Show which point lies on the curve and which does not.

| | | | |
|---|---|---|---|
| For (1,2): | $(1)^4 + (2)^3 = 9$ | $=$ | $9 = 2 + \frac{7}{2}(1)(2)$ |
| For (3,4): | $(3)^4 + (4)^3 = 145$ | $\neq$ | $44 = 2 + \frac{7}{2}(3)(4)$ |

Thus (1,2) is on the curve but (3,4) is not.

(d) (4 points) Find an equation of the tangent line to the curve at the point from part (c) that is on the curve.

For the point (1,2), we see that $\frac{dy}{dx} = \frac{6}{17}$. Therefore the equation of the tangent line to the curve at (1,2), using the point-slope formula, is given by

$$y - 2 = \frac{6}{17}(x - 1)$$

Or, in more standard form,

$$y = \frac{6}{17}x + \frac{28}{17}$$