6. (12 points) It's time to redesign the layout of exhibits at the San Diego Zoo. The zookeeper, Joan Embery, has been told by Paco Underhill that more exhibits will attract more visitors to enter the zoo but, as the space between exhibits decreases, more zoo visitors are likely to brush butts and flee the zoo in disgust.[1]

   Paco has modeled the predicted number of visitors to the zoo each year, in millions of people, by the function

$$f(x) = axe^{-bx} + c$$

   where x represents the number of exhibits per acre.

   Since the park is beautiful on its own, Paco believes that 1/2 million visitors a year will come to the area, even if there are no exhibits. He has determined that the maximum number of visitors will come to the zoo if there are 4 exhibits per acre. (After that, the "disgust factor" begins to creep in.) According to Paco's model, when the number of exhibits per acre is 4, the number of visitors would be approximately 2.5 million people per year. Use this information (and calculus) to solve for a, b and c in the above function.

   [Note: The maximum number of exhibits that could be packed into an acre of land is 8, since exhibits require 500 m^2, on average, and an acre $\approx$ 4000 m^2.]

Since half a million visitors will visit the zoo even if there are no exhibits, we have $f(0) = c = \frac{1}{2}$ million visitors. Because the maximum number of visitors occurs when $x = 4$, we know that $f'(4) = 0$:

$$\begin{aligned} f'(x) &= ae^{-bx} - abxe^{-bx}, \\ f'(4) &= ae^{-4b} - 4abe^{-4b} \\ &= ae^{-4b}(1 - 4b). \end{aligned}$$

Thus, if $f'(4) = 0$, we must have $1 - 4b = 0$, so $b = \dfrac{1}{4}$.

Because the maximum number of visitors is 2.5 million, we know that $f(4) = 2.5$ (million visitors). Thus

$$2.5 = f(4) = 4ae^{-\frac{1}{4}4} + \frac{1}{2}$$

Solving for a, we find that $a = \dfrac{e}{2}$.

[Note: either first or second derivative test should be used to verify that a local maximum DOES occur at $x = 4$–or the endpoints of the interval [0,8] could be used to show that 4 exhibits gives the maximum.]

$a =$ $\dfrac{e}{2}$

$b =$ $\dfrac{1}{4}$

$c =$ $\dfrac{1}{2}$

[1] See `http://en.wikipedia.org/wiki/Joan_Embery`. and `http://www.amazon.com/Why-We-Buy-Science-Shopping/dp/0684849143`. When he finishes with the zoo, Underhill will arrange the classroom tables in Dennison Hall.