3. (10 points) Suppose \( f'(x) \) is a differentiable increasing function for all \( x \). In each of the following pairs, circle the larger value. In each case, give a brief reason for your choice. (Assume that none of the values below are equal for this function and \( \Delta x \neq 0 \)).

(a) \( f'(5) \) and \( f'(6) \)

\( f' \) is increasing, and \( 5 < 6 \).

(b) \( f''(5) \) and 0

\( f' \) is increasing, so its derivative \( f'' \) is always positive (since \( f''(5) \neq 0 \)).

(c) \( f(5 + \Delta x) \) and \( f(5) + f'(5)\Delta x \)

Since \( f''(5) > 0 \), \( f \) is concave up at 5. Therefore its values near 5 are greater than the linear approximation, which is \( f(5) + f'(5)\Delta x \).

4. (15 points) Using calculus, find constants \( a \) and \( b \) in the function \( f(x) = axe^{bx} \) such that \( f\left(\frac{1}{3}\right) = 1 \) and the function has a local maximum at \( x = \frac{1}{3} \). Once you have found \( a \) and \( b \), verify that your answer satisfies the given conditions. Show all work.

Since \( f\left(\frac{1}{3}\right) = 1 \), we have an equation \( 1 = a \left(\frac{1}{3}\right) e^{b/3} \), so solving for \( a \) gives

\[
a = 3e^{-b/3}.
\]

Now for \( f \) to have a local maximum at \( \frac{1}{3} \), we want \( f'(\frac{1}{3}) = 0 \) and \( f''(\frac{1}{3}) < 0 \). Let’s solve the first equation, using our expression for \( a \).

\[
f'(x) = ae^{bx} + abxe^{bx} = 3e^{-b/3}e^{bx} + 3bxe^{-b/3}e^{bx} ;
\]

\[
f'(\frac{1}{3}) = 3e^{-b/3}e^{b/3} + be^{-b/3}e^{b/3} = 3 + b,
\]

and setting this equal to zero tells us \( b = -3 \). Plugging this into the expression for \( a \), we get \( a = 3e \).

Here, verify that \( f\left(\frac{1}{3}\right) = 1 \) and \( f'(\frac{1}{3}) = 0 \). However, most important, check the critical point to see that it is a local maximum. We use the second derivative test, and check \( f''(\frac{1}{3}) < 0 \):

\[
f''(x) = -9ee^{-3x} - 9ee^{-3x} + 27exe^{-3x} = -18ee^{-3x} + 27exe^{-3x} ;
\]

\[
f''\left(\frac{1}{3}\right) = -18ee^{-1} + 27e\left(\frac{1}{3}\right) e^{-1} = -18 + 9 = -9 < 0.
\]

Thus, the critical point at \( x = \frac{1}{3} \) is a local maximum.