3. (10 points) Suppose $f'(x)$ is a differentiable increasing function for all $x$. In each of the following pairs, circle the larger value. In each case, give a **brief** reason for your choice.(Assume that none of the values below are equal for this function and $\Delta x \neq 0$).

(a) $f'(5)$ and $\boxed{f'(6)}$

$f'$ is increasing, and $5 < 6$.

(b) $\boxed{f''(5)}$ and $0$

$f'$ is increasing, so its derivative $f''$ is always positive (since $f''(5) \neq 0$) .

(c) $\boxed{f(5+\Delta x)}$ and $f(5) + f'(5)\Delta x$

Since $f''(5) > 0$, $f$ is concave up at 5. Therefore its values near 5 are greater than the linear approximation, which is $f(5) + f'(5)\Delta x$.

4. (15 points) Using calculus, find constants $a$ and $b$ in the function $f(x) = axe^{bx}$ such that $f(\frac{1}{3}) = 1$ and the function has a local maximum at $x = \frac{1}{3}$. Once you have found $a$ and $b$, verify that your answer satisfies the given conditions. Show all work.

Since $f(\frac{1}{3}) = 1$, we have an equation $1 = a\left(\frac{1}{3}\right)e^{b/3}$, so solving for $a$ gives

$$a = 3e^{-b/3}.$$

Now for $f$ to have a local maximum at $\frac{1}{3}$, we want $f'(\frac{1}{3}) = 0$ and $f''(\frac{1}{3}) < 0$. Let's solve the first equation, using our expression for $a$.

$$\begin{aligned} f'(x) &= ae^{bx} + abxe^{bx} \\ &= 3e^{-b/3}e^{bx} + 3bxe^{-b/3}e^{bx}, \\ f'\left(\frac{1}{3}\right) &= 3e^{-b/3}e^{b/3} + be^{-b/3}e^{b/3} \\ &= 3 + b, \end{aligned}$$

and setting this equal to zero tells us $b = -3$. Plugging this into the expression for $a$, we get $a = 3e$.

Here, verify that $f(\frac{1}{3}) = 1$ and $f'(\frac{1}{3}) = 0$. However, most important, check the critical point to see that it is a local maximum. We use the second derivative test, and check $f''(\frac{1}{3}) < 0$:

$$\begin{aligned} f''(x) &= -9ee^{-3x} - 9ee^{-3x} + 27exe^{-3x} \\ &= -18ee^{-3x} + 27exe^{-3x}; \\ f''\left(\frac{1}{3}\right) &= -18ee^{-1} + 27e\left(\frac{1}{3}\right)e^{-1} \\ &= -18 + 9 \\ &= -9 < 0. \end{aligned}$$

Thus, the critical point at $x = \frac{1}{3}$ is a local maximum.