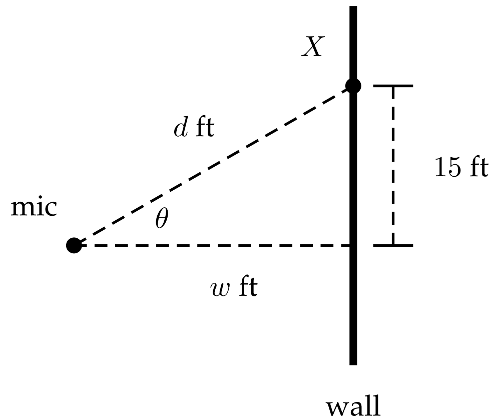
5. (16 points) A directional microphone is mounted on a stand facing a wall. The sensitivity S of the microphone to sounds at point X on the wall is inversely proportional to the square of the distance d from the point X to the mic, and directly proportional to the cosine of the angle θ. That is, $S = K\frac{\cos\theta}{d^2}$ for some constant K. (See the diagram below.) How far from the wall should the mic be placed to maximize sensitivity to sounds at X?



We are given that $S = K\frac{\cos\theta}{d^2}$. Also, from the definition of cosine we see $\cos\theta = \frac{w}{d}$. By the Pythagorean theorem, $d^2 = w^2 + 15^2 = w^2 + 225$. Therefore

$$S = K\frac{\cos\theta}{d^2} = K\frac{\frac{w}{d}}{w^2+225} = K\frac{w}{(w^2+225)^{3/2}}.$$

Differentiating, we get

$$\frac{dS}{dw} = K\frac{(w^2+225)^{3/2} - 3w^2(w^2+225)^{1/2}}{(w^2+225)^3}.$$

The derivative is defined for all w and is only equal to zero when the numerator is zero. Factoring the common factor of $(w^2+225)^{1/2}$ gives

$$(w^2+225)^{1/2}(w^2+225-3w^2),$$

and since (w^2+225) is never zero, we must have

$$2w^2 = 225,$$

or

$$w = \pm\sqrt{\frac{225}{2}} = \pm\frac{15}{\sqrt{2}}.$$

Since w is a length, we discard the negative root, and now must test the one critical point $w = \frac{15}{\sqrt{2}}$. Note that for $w < \frac{15}{\sqrt{2}}$, the first derivative is positive, and for $w > \frac{15}{\sqrt{2}}$ the derivative is negative. Thus, by the first derivative test, $w = \frac{15}{\sqrt{2}}$ is a local maximum. Since the function is continuous and this is the only critical point on the domain, $w = \frac{15}{\sqrt{2}}$ is the global maximum. The mic should be placed $\frac{15}{\sqrt{2}} \approx 10.6$ feet from the wall to maximize the sensitivity to sounds at X.