7. (a) (4 points) Show that the point $(x, y) = (3, -6)$ lies on the curve defined by $y^2 - x^3 - x^2 = 0$.

Check: $(-6)^2 - 3^3 - 3^2 = 36 - 27 - 9 = 0$.

(b) (4 points) What is the equation of the tangent line to the curve at the point $(3, -6)$?

Use implicit differentiation to get a formula for $\frac{dy}{dx}$:

$$2y\frac{dy}{dx} - 3x^2 - 2x = 0,$$

so

$$\frac{dy}{dx} = \frac{3x^2 + 2x}{2y}.$$

Evaluating at $(x, y) = (3, -6)$, we get $\frac{dy}{dx} = \frac{27+6}{-12} = -\frac{33}{12}$. This is the slope of the tangent line, so its equation is

$$y = -\frac{33}{12}(x - 3) - 6.$$

(c) (2 points) Consider the function $f(x) = x\sqrt{x+1}$. What is the domain of f?

The function f is defined whenever $\sqrt{x+1}$ is defined, that is, when $x + 1 \geq 0$, or $x \geq -1$. [Note that f is one of the explicitly defined functions for the curve above by solving for y and restricting y to the positive root.]

(d) (6 points) Find all critical points, local maxima, and local minima of f. Which of the local maxima and minima are global maxima / minima?

First, take the derivative:

$$\begin{aligned} f'(x) &= \sqrt{x+1} + \frac{1}{2}\frac{x}{\sqrt{x+1}} \\ &= \frac{2(x+1) + x}{2\sqrt{x+1}} \\ &= \frac{3x+2}{2\sqrt{x+1}}. \end{aligned}$$

The derivative is 0 when $3x + 2 = 0$, *i.e.*, when $x = -\frac{2}{3}$. The derivative is undefined for $x = -1$, which is in the domain of f.

Thus the critical points are $x = -\frac{2}{3}$ and $x = -1$. To check if these are local maxima or minima, we can apply the first derivative test. Near $-\frac{2}{3}$, we have $3x + 2 < 0$ if $x < -\frac{2}{3}$ and $3x + 2 > 0$ if $x > -\frac{2}{3}$. Since $f'(x)$ has the same sign as $3x + 2$, we conclude that f has a local minimum at $x = -\frac{2}{3}$. As for the endpoint -1, for $x > -1$ but near -1, we have $3x + 2 < 0$, so $f'(x) < 0$. Therefore f has a local maximum at $x = -1$. Note that $f(x) \to \infty$ as $x \to \infty$, so the local minimum at $x = -\frac{2}{3}$ is a global minimum. There is no global maximum.