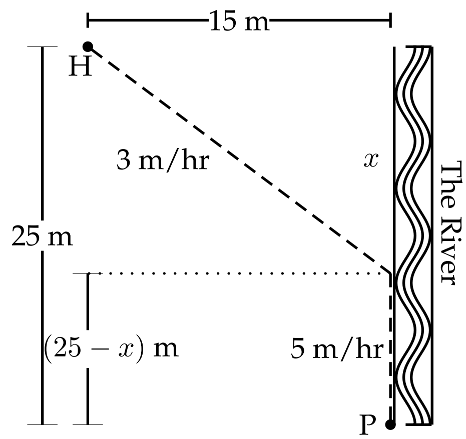
4. (16 points) The Awkward Turtle is going to a dinner party! Unfortunately, he's running quite late, so he wants to take the quickest route. The Awkward Turtle lives in a grassy plain (his home is labeled H in the figure below), where his walking speed is a slow but steady 3 meters per hour. The party is taking place southeast of his home, on the bank of a river (denoted by P in the figure). The river flows south at a constant rate of 5 meters per hour, and once he gets to the river, the Awkward Turtle can jump in and float the rest of the way to the party on his back. A typical path the Awkward Turtle might take from his house to the party is indicated in the figure below by a dashed line.

   What is the shortest amount of time the entire trip (from home to dinner party) can take? [*Recall that rate* $\times$ *time* = *distance.*]



Let $t(x)$ denote the amount of time the trip takes if the Awkward Turtle floats along the river for $(25-x)$ meters (see the picture above). By the Pythagorean theorem, the distance the turtle will walk across the grassy plain is $\sqrt{15^2+x^2}$; therefore, we have

$$t(x) = \frac{1}{3}\sqrt{15^2+x^2} + \frac{25-x}{5}.$$

To minimize this, we take the derivative and set it equal to 0. A computation shows that

$$t'(x) = \frac{1}{3} \cdot \frac{x}{\sqrt{15^2+x^2}} - \frac{1}{5}.$$

Setting this equal to 0 and solving yields $x = \frac{45}{4} = 11.25$ m. Thus, such a trip takes $t(11.25) = 9$ hours.

Using the quotient rule we see that

$$t''(x) = \frac{1}{3}\left(\frac{\sqrt{15^2+x^2} - \frac{x^2}{\sqrt{15^2+x^2}}}{15^2+x^2}\right) = \frac{1}{3}\left(\frac{15^2}{\left(\sqrt{15^2+x^2}\right)^3}\right)$$

Since $t''(x) > 0$ for all x, we have that 9 hours is a local minimum. To determine whether $t = 9$ is the global minimum, we can either show that since there is only one critical point the local minimum is the global minimum or check the endpoints. The least the turtle can float along the river is 0 meters, in which case the trip takes $t(0) = 10$ hours; the greatest distance he can float is 25 meters, in which case the trip would take $t(25) \approx 9.718$ hours. Therefore, 9 hours is indeed the global minimum. (Alternatively, one can see this from the graph the function $y = t(x)$.)

Minimal time = 9 hours