9. (a) (4 points) Suppose that the tangent line to the function $y = f(x)$ at $x = c$ passes through the origin. Express $\left.\frac{dy}{dx}\right|_{x=c}$ in terms of c and $f(c)$.

> We are given that the line passes through the origin and the point $\big(c, f(c)\big)$, so its slope is $\frac{f(c)}{c}$. On the other hand, by definition its slope is $\left.\frac{dy}{dx}\right|_{x=c}$. Therefore,
>
> $$\left.\frac{dy}{dx}\right|_{x=c} = \frac{f(c)}{c}.$$

(b) (6 points) Consider the graph of $xy = ae^{by}$, where both a and b are positive (non-zero) constants. Determine $\frac{dy}{dx}$.

> Differentiating both sides of the equation implicitly, we find
>
> $$y + xy' = ae^{by} \cdot by'$$
>
> Solving for y' we find
>
> $$\frac{dy}{dx} = \frac{y}{abe^{by} - x}.$$

(c) (6 points) Write down the equations of all lines passing through the origin which are tangent to the curve $xy = ae^{by}$, where as before a and b are positive (nonzero) constants. [*Hint: You may find it helpful to rewrite your answer to 9b without exponentials, by using substitution – by the definition of the curve, you can replace the quantity* ae^{by} *by* xy.]

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Following the hint, we rewrite the answer to (b) in the form

$$\frac{dy}{dx} = \frac{y}{bxy - x}.$$

Suppose that a line through the origin is tangent to the curve at the point (x_0, y_0). Note that neither x_0 nor y_0 can be 0; otherwise, from the equation of the curve we would have $0 = ae^{by_0}$, which would contradict the positivity of a.

By the same reasoning as in part (a),

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{y_0}{x_0}.$$

On the other hand, from above,

$$\left.\frac{dy}{dx}\right|_{(x_0,y_0)} = \frac{y_0}{bx_0y_0 - x_0}.$$

Therefore, $\frac{y_0}{x_0} = \frac{y_0}{bx_0y_0 - x_0}$. Dividing both sides of the equation by $\frac{y_0}{x_0}$ (this is OK since y_0 is not zero), we obtain

$$1 = \frac{1}{by_0 - 1}.$$

This immediately implies that $y_0 = \frac{2}{b}$, from which (using the equation defining the curve) we deduce that $x_0 = \frac{ab}{2}e^2$.

Thus, the slope of the line in question is $\frac{y_0}{x_0} = \frac{4}{ab^2e^2}$. So the equation of the line is

$$y = \frac{4}{ab^2e^2}x.$$