5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

Solution: We use implicit differentiation:

$$2x - 2y\frac{dy}{dx} = 2 + \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + \frac{dy}{dx} + 0.$$

Then solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2x - 2 - y &= \frac{dy}{dx}(x + 2y + 1) \\ \frac{dy}{dx} &= \frac{2x - 2 - y}{x + 2y + 1} \end{aligned}$$

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

Solution: For the point $(4, 2)$, $x^2 - y^2 = 4^2 - 2^2 = 12$ and $2x + xy + y + 2 = 2(4) + 4(2) + 2 + 2 = 20$ are not equal, so $(4, 2)$ IS NOT on the hyperbola.

For the point $(2, -1)$, $x^2 - y^2 = 2^2 - (-1)^2 = 3$ and $2x + xy + y + 2 = 2(2) + 2(-1) - 1 + 2 = 3$ are equal, so $(2, -1)$ IS on the hyperbola.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution: From part (a),

$$\frac{dy}{dx} = \frac{2x - 2 - y}{x + 2y + 1},$$

so

$$\frac{dy}{dx}\Big|_{(x,y)=(2,-1)} = \frac{2(2) - 2 - (-1)}{2 + 2(-1) + 1} = \frac{3}{1} = 3.$$

Then the equation of the tangent line is $y = 3(x - 2) - 1$ or $y = 3x - 7$.