4. [15 points] A model for the amount of an antihistamine in the bloodstream after a patient takes a dose of the drug gives the amount, a, as a function of time, t, to be $a(t) = A(e^{-t} - e^{-kt})$. In this equation, A is a measure of the dose of antihistamine given to the patient, and k is a transfer rate between the gastrointestinal tract and the bloodstream. A and k are positive constants, and for pharmaceuticals like antihistamine, $k > 1$.

a. [5 points] Find the location $t = T_m$ of the non-zero critical point of $a(t)$.

Solution: The maximum will occur at an endpoint or at a critical point, when $a'(t) = 0$. The critical points are thus where $a'(t) = A(-e^{-t} + ke^{-kt}) = 0$. Solving, we have $e^{-t} = ke^{-kt}$, so that $e^{(k-1)t} = k$, or $t = T_m = \frac{1}{k-1}\ln(k)$.

b. [3 points] Explain why $t = T_m$ is a global maximum of $a(t)$ by referring to the expression for $a(t)$ or $a'(t)$.

Solution: Note that $a'(0) = A(k-1) > 0$ and that for large t, $a'(t) = A(-e^{-t}+ke^{-kt}) < 0$ ($k > 1$ guarantees that the second exponential decays much faster than the first). Thus the critical point must be a maximum. In addition, because $t = T_m$ this is the only critical point we know it must be the global maximum.
Alternately, note that $a(0) = 0$. Because $k > 0$, $a(t) \geq 0$ for all t (the exponential involving $-kt$ will decay faster than e^{-t}). And for large t, $a(t) \to 0$. Thus at $t = T_m$, $a(t)$ must take on a maximum value, and because it is the only critical point this must be the global maximum.

c. [4 points] The function $a(t)$ has a single inflection point. Find the location $t = T_I$ of this inflection point. You do not need to prove that this is an inflection point.

Solution: To find inflection points, we look for where $a''(t) = 0$. This gives $a''(t) = A(e^{-t} - k^2e^{-kt}) = 0$. Solving for t, we have (similarly to in (a)) $e^{(k-1)t} = k^2$, so that $t = T_I = \frac{1}{k-1}\ln(k^2) = \frac{2}{k-1}\ln(k)$.
(We could show that this is an inflection point by a similar argument to (b): because $k > 1$ we know that $a''(0) < 0$ and as t increases $a''(t)$ must eventually become positive. Thus this is an inflection point.)

d. [3 points] Using your expression for T_m from (a), find the rate at which T_m changes as k changes.

Solution: We have $T_m = \frac{1}{k-1}\ln(k)$. Thus

$$\frac{dT_m}{dk} = -\frac{1}{(k-1)^2}\ln(k) + \frac{1}{k(k-1)}.$$