3. [12 points] The following questions relate to the implicit function

$$y^2 + 4x = 4xy^2.$$

a. [4 points] Compute $\frac{dy}{dx}$.

Solution: Differentiating the equation with respect to x, we have

$$2y\frac{dy}{dx} + 4 = 4y^2 + 8xy\frac{dy}{dx}.$$

Gathering terms involving $\frac{dy}{dx}$ to one side, the equation becomes

$$2y\frac{dy}{dx} - 8xy\frac{dy}{dx} = 4y^2 - 4$$

which gives the solution

$$\frac{dy}{dx} = \frac{4y^2 - 4}{2y - 8xy}.$$

b. [4 points] Find the equation for the tangent line to this curve at the point $(\frac{1}{3}, 2)$.

Solution: The slope is

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{3},2)} = \frac{4 \cdot 2^2 - 4}{2 \cdot 2 - 8 \cdot \frac{1}{3} \cdot 2} = -9,$$

so by the point-slope formula, the equation is

$$y = -9x + 5.$$

c. [4 points] Find the x- and y-coordinates of all points at which the tangent line to this curve is vertical.

Solution: The slope is undefined as these points, so we must have $2y - 8xy = 0$. Factoring out a $2y$ we get

$$2y(1 - 4x) = 0$$

which gives the solutions $y = 0$ or $x = \frac{1}{4}$. Plugging into the equation for the implicit function, $y = 0$ gives the point $(0, 0)$. However, when we plug in $x = \frac{1}{4}$, we get the equation $y^2 + 1 = y^2$, which has no solutions. Therefore, $(0, 0)$ is the only point at which the tangent line is vertical.