4. [12 points] Consider the family of functions
\[ f(x) = ax - e^{bx}, \]
where \(a\) and \(b\) are positive constants.

a. [4 points] Any function \(f(x)\) in this family has only one critical point. In terms of \(a\) and \(b\), what are the \(x\)- and \(y\)-coordinates of that critical point?

Solution: We find
\[ f'(x) = a - be^{bx}, \]
so setting this equal to zero and solving for \(x\) shows that there is a critical point at
\[ x = \frac{1}{b} \ln \left( \frac{a}{b} \right). \]
The \(y\)-coordinate of the critical point is
\[ y = f \left( \frac{1}{b} \ln \left( \frac{a}{b} \right) \right) = \frac{a}{b} \ln \left( \frac{a}{b} \right) - \frac{a}{b}. \]

b. [4 points] Is the critical point a local maximum or a local minimum? Justify your answer with either the first-derivative test or the second-derivative test.

Solution: The second derivative of \(f(x)\) is \(f''(x) = -b^2 e^{bx}\), so
\[ f'' \left( \frac{1}{b} \ln \left( \frac{a}{b} \right) \right) = -b^2 \cdot \frac{a}{b} = -ab, \]
which is negative since \(a\) and \(b\) are both positive. Therefore, the second derivative test tells us that the critical point is a local maximum.

c. [4 points] For which values of \(a\) and \(b\) will \(f(x)\) have a critical point at \((1,0)\)?

Solution: We need
\[ \frac{1}{b} \ln \left( \frac{a}{b} \right) = 1 \quad \text{and} \quad \frac{a}{b} \ln \left( \frac{a}{b} \right) - \frac{a}{b} = 0. \]
The first equation rearranges to \(\ln(\frac{a}{b}) = b\), and if we plug this into the second equation, we obtain
\[ a - \frac{a}{b} = 0 \Rightarrow 1 - \frac{1}{b} = 0 \Rightarrow b = 1. \]
Plugging this into either equation and solving for \(a\) gives
\[ a = e. \]