4. [12 points] Consider the family of functions

$$f(x) = ax - e^{bx},$$

where a and b are positive constants.

a. [4 points] Any function $f(x)$ in this family has only one critical point. In terms of a and b, what are the x- and y-coordinates of that critical point?

Solution: We find

$$f'(x) = a - be^{bx},$$

so setting this equal to zero and solving for x shows that there is a critical point at

$$x = \frac{1}{b}\ln\left(\frac{a}{b}\right).$$

The y-coordinate of the critical point is

$$y = f\left(\frac{1}{b}\ln\left(\frac{a}{b}\right)\right) = \frac{a}{b}\ln\left(\frac{a}{b}\right) - \frac{a}{b}.$$

b. [4 points] Is the critical point a local maximum or a local minimum? Justify your answer with either the first-derivative test or the second-derivative test.

Solution: The second derivative of $f(x)$ is $f''(x) = -b^2e^{bx}$, so

$$f''\left(\frac{1}{b}\ln\frac{a}{b}\right) = -b^2 \cdot \frac{a}{b} = -ab,$$

which is negative since a and b are both positive. Therefore, the second derivative test tells us that the critical point is a local maximum.

c. [4 points] For which values of a and b will $f(x)$ have a critical point at $(1, 0)$?

Solution: We need

$$\frac{1}{b}\ln\frac{a}{b} = 1 \quad \textbf{and} \quad \frac{a}{b}\ln\frac{a}{b} - \frac{a}{b} = 0.$$

The first equation rearranges to $\ln(\frac{a}{b}) = b$, and if we plug this into the second equation, we obtain

$$a - \frac{a}{b} = 0 \Rightarrow 1 - \frac{1}{b} = 0 \Rightarrow b = 1.$$

Plugging this into either equation and solving for a gives

$$a = e.$$