5. [12 points]

a. [3 points] Find the local linearization $L(x)$ of the function

$$f(x) = (1+x)^k$$

near $x = 0$, where k is a positive constant.

Solution: The derivative is $f'(x) = k(1+x)^{k-1}$, so the slope of the tangent line at $x = 0$ is

$$f'(0) = k.$$

Since $f(0) = 1^k = 1$, the tangent line passes through the point $(0,1)$. Therefore, the point-slope formula shows that the equation of the tangent line is

$$y = kx + 1.$$

b. [3 points] For which values of k does this local linearization give underestimates of the actual value of $f(x)$? (Show your work.)

Solution: The local linearization gives underestimates of the actual value when $f''(0) > 0$. The second derivative is $f''(x) = k(k-1)(1+x)^{k-2}$, so

$$f''(0) = k(k-1).$$

Since $k > 0$, this is positive when the second factor is positive, which is when $k > 1$.

c. [2 points] Suppose you want to use $L(x)$ to find an approximation of the number $\sqrt{1.1}$. What number should k be, and what number should x be?

Solution: If $k = \frac{1}{2}$ and $x = 0.1$, then $f(0.1) = \sqrt{1.1}$, so $L(1.1)$ gives an approximation of $\sqrt{1.1}$.

d. [2 points] Approximate $\sqrt{1.1}$ using $L(x)$.

Solution: If k and x are as above, then $\sqrt{1.1} \approx L(0.1) = 1.05$.

e. [2 points] What is the error in the approximation from part (d)?

Solution: The error is the actual value minus the approximate value which is $\sqrt{1.1} - 1.05 \approx -0.00119$.