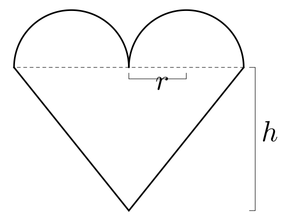
7. [12 points] For Valentine's Day, Jason decides to make a heart-shaped cookie for Sophie to try to win her over. Being mathematically-minded, the only kind of heart that Jason knows how to construct is composed of two half-circles of radius r and an isosceles triangle of height h, as shown below.

Jason happens to know that Sophie's love for him will be determined by the dimensions of the cookie she receives; if given a cookie as described above, her love L will be

$$L = hr^2,$$

where r and h are measured in centimeters and L is measured in pitter-patters, a standard unit of affection. If Jason wants to make a cookie whose area is exactly 300cm^2, what should the dimensions be to maximize Sophie's love?



Solution: The area of the heart shape is

$$A = \pi r^2 + 2rh.$$

Setting this equal to 300 and solving for h gives the formula

$$h = \frac{300 - \pi r^2}{2r} = 150r^{-1} - \frac{\pi}{2}r.$$

Therefore, the formula for L can be written in terms of r alone:

$$L(r) = (150r^{-1} - \tfrac{\pi}{2}r)r^2 = 150r - \tfrac{\pi}{2}r^3.$$

We need to find the global maximum of $L(r)$. We have

$$L'(r) = 150 - \frac{3\pi}{2}r^2 = 0 \Rightarrow r = \frac{10}{\sqrt{\pi}}.$$

This critical point is a local maximum of L by the second-derivative test, since

$$L''(r) = -3\pi r \Rightarrow L''(10/\sqrt{\pi}) = -30\sqrt{\pi} < 0.$$

Since it is the only critical point, it must therefore be the global maximum.
Plugging this value of r into our formula, we can find the value of h, as well. We find that the dimensions that maximize Sophie's love are:

$$r = 10/\sqrt{\pi} \approx 5.6419\text{cm},$$

$$h = 10\sqrt{\pi} \approx 17.7245\text{cm}.$$