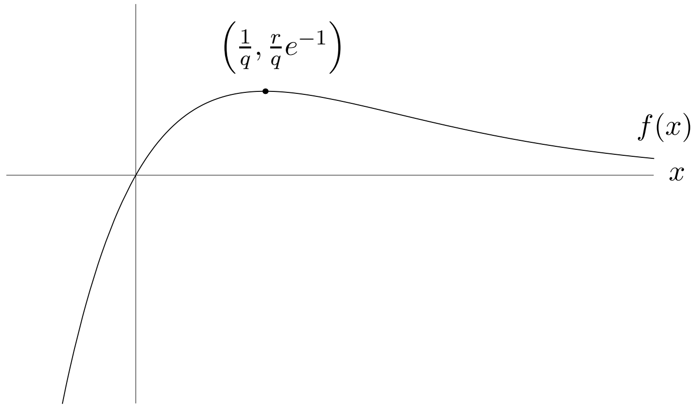
8. [16 points] Below is the graph of the function

$$f(x) = rxe^{-qx},$$

where r and q are constants. Assume that both r and q are greater than 1. The function $f(x)$ passes through the origin and has a local maximum at the point $P = \left(\frac{1}{q}, \frac{r}{q}e^{-1}\right)$, as shown in the graph.



a. [4 points] Justify, using either the first-derivative test or second-derivative test, that the point P is a local maximum.

Solution: To apply the first-derivative test, first compute:

$$f'(x) = re^{-qx}(-qx + 1).$$

Thus, there is indeed a critical point at $x = \frac{1}{q}$. Plugging in $x = 0$ (which is less than $\frac{1}{q}$), we find $f'(0) = r > 0$, while plugging in $x = \frac{2}{q}$ (which is greater than $\frac{1}{q}$), we find $f'(\frac{2}{q}) = -re^{-2} < 0$. Thus, f' changes from positive to negative at $x = \frac{1}{q}$, so it is a local maximum.
To apply the second-derivative test, compute $f''(x) = -rqe^{-qx}(-qx + 2)$, so

$$f''(\tfrac{1}{q}) = -rqe^{-1} < 0.$$

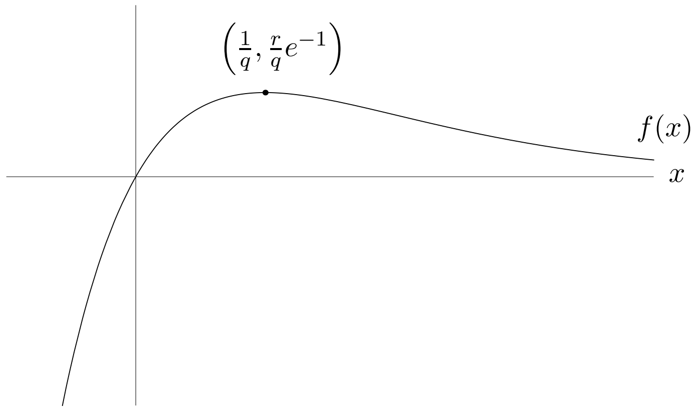
b. [2 points] What are the x-coordinates of the global maximum and minimum of $f(x)$ on the domain $[0, 1]$? (If $f(x)$ does not have a global maximum on this domain, say "no global maximum", and similarly if $f(x)$ does not have a global minimum.)

Solution: Since $q > 1$, the local maximum at $x = \frac{1}{q}$ is within this domain. Therefore, the global maximum occurs at $x = \frac{1}{q}$ and the global minimum occurs at $x = 0$.

c. [2 points] What are the x-coordinates of the global maximum and minimum of $f(x)$ on the domain $(-\infty, \infty)$? (If $f(x)$ does not have a global maximum on this domain, say "no global maximum", and similarly if $f(x)$ does not have a global minimum.)

Solution: The global maximum is at $x = \frac{1}{q}$ and there is no global minimum.

8. (continued) For your convenience, the graph of $f(x)$ is repeated below.



d. [4 points] Suppose that $g(x)$ is a function with $g'(x) = f(x)$. Find x-values of all local maxima and minima of $g(x)$. Justify that each maximum you find is a maximum and each minimum is a minimum.

Solution: The function $g(x)$ has critical points wherever $g'(x) = f(x) = 0$, which is only at $x = 0$. Since $f(x)$ changes from negative to positive at this point, the critical point at $x = 0$ is a local minimum by the first-derivative test.

e. [4 points] If $g(x)$ is as in part (d), for which x-values does $g(x)$ have inflection points? Show that these x-values are indeed inflection points.

Solution: The function $g(x)$ has inflection points when $g''(x) = f'(x)$ changes sign. This occurs precisely when $f(x)$ changes from increasing to decreasing (or vice versa), which is at $x = \frac{1}{q}$.