5. [14 points] Consider the family of functions

$$g(x) = \frac{ax^b}{\ln(x)}$$

where a and b are nonzero constants.

a. [4 points] Calculate $g'(x)$.

Solution: Using the quotient rule, we get

$$g'(x) = \frac{abx^{b-1}\ln(x) - ax^{b-1}}{(\ln(x))^2}.$$

b. [6 points] Find values for a and b so that $g(e) = 1$ and $g'(e) = 0$.

Solution: Since $g(e) = 1$ we have

$$ae^b = 1. \qquad (\star)$$

Since $g'(e) = 0$, the numerator of the answer to (a) must be zero, which says

$$abe^{b-1} - ae^{b-1}. \qquad (\dagger)$$

The equation ($\dagger$) simplifies to $b = 1$, and then from ($\star$) we deduce $a = \frac{1}{e}$.

c. [4 points] With the values of a and b you found in (b), is $x = e$ a local minimum of g, a local maximum of g or neither? Justify your answer.

Solution: It is a local minimum. To see this, we use the first derivative test. The denominator of our expression for g' is always positive, and (with our values of a and b) the numerator is

$$\frac{1}{e}(\ln(x) - 1).$$

This expression changes signs from negative to positive around $x = e$.