10. [10 points] Let $f(x)$ be a function with $f(1) = 5$, $f'(1) = -2$, and $f''(1) = 3$.

a. [2 points] Use the local linearization of $f(x)$ at $x = 1$ to estimate $f(0.9)$.

Solution: The local linearization of $f(x)$ at $x = 1$ is $5 - 2(x - 1)$. Plugging in $x = 0.9$ yields $5 - 2(0.9 - 1) = 5 + 0.2 = 5.2$.

Answer: $f(0.9) \approx$ 5.2

b. [2 points] Do you expect your estimate from Part (a) to be an overestimate or underestimate? To receive any credit on this question, you must justify your answer.

Solution: Since $f''(1) = 3$, the graph of $y = f(x)$ is concave up near $x = 1$. Therefore, the tangent line at $x = 1$ lies under the graph of $f(x)$ near $x = 1$, so we expect this to be an underestimate.

c. [2 points] Use the tangent line approximation of $f'(x)$ near $x = 1$ to estimate $f'(1.1)$.

Solution: The tangent line to $f'(x)$ at $x = 1$ passes through the point $(1, f'(1))$ and has slope $f''(1)$ (as the slope of the derivative function is given by the second derivative). Therefore, the tangent line to $f'(x)$ at $x = 1$ is given by the equation

$$L = -2 + 3(x - 1).$$

Plugging in $x = 1.1$ yields $-2 + 3(1.1 - 1) = -1.7$.

Answer: $f'(1.1) \approx$ -1.7

d. [4 points] Suppose that the tangent line approximation of $f(x)$ near $x = 8$ estimates $f(8.05)$ to be 3.75 and $f(8.1)$ to be 3.6. Find $f(8)$ and $f'(8)$.

Solution: Since the tangent line passes through the points $(8.05, 3.75)$ and $(8.1, 3.6)$, it has slope

$$\frac{3.6 - 3.75}{8.1 - 8.05} = \frac{-0.15}{0.05} = -3.$$

Hence $f'(8) = -3$. Moreover, an equation for this tangent line is therefore

$$L = 3.75 - 3(x - 8.05),$$

so plugging in $x = 8$, it passes through the point $(8, 3.9)$.
By definition of the tangent line, then, we have that $f(x)$ also passes through the point $(8, 3.9)$ and also has slope -3, so we conclude that $f(8) = 3.9$ and $f'(8) = -3$.

Answer: $f(8) =$ 3.9 and $f'(8) =$ -3