5. [13 points] Suppose $f(x)$ is a function defined for all x whose <u>derivative</u> and <u>second derivative</u> are given by

$$f'(x) = \frac{(x+2)^2(x-3)}{(x+1)^{1/3}} \quad \text{and} \quad f''(x) = \frac{2(x+2)(x-1)(4x+3)}{3(x+1)^{4/3}}.$$

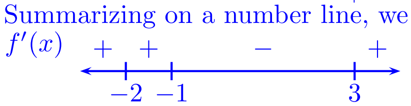
a. [2 points] Find the x-coordinates of all critical points of $f(x)$. If there are none, write NONE.

Solution: Critical points of $f(x)$ occur where $f'(x)$ is zero or undefined. $f'(x)$ is zero when the numerator is zero, at $x = -2$ and $x = 3$. $f'(x)$ is undefined when the denominator is zero, at $x = -1$. Therefore, $x = -2, -1, 3$ are the critical points of $f(x)$.

Answer: Critical point(s) at $x =$ $-2, -1, 3$

b. [6 points] Find the x-coordinates of all local extrema of $f(x)$.
If there are none of a particular type, write NONE.
Justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: To classify the critical points, we use the First Derivative Test, so we look at the sign of $f'(x)$ before and after each critical point.
For $x < -2$, $(x+2)$, $(x-3)$, and $(x+1)$ are all negative, so $f'(x) = \frac{-^2 \cdot -}{-} = +$ is positive.
Similarly, for $-2 < x < -1$, we have $f'(x) = \frac{+^2 \cdot -}{-} = +$.
For $-1 < x < 3$, we have $f'(x) = \frac{+^2 \cdot -}{+} = -$.
For $x > 3$, we have $f'(x) = \frac{+^2 \cdot +}{+} = +$.
Summarizing on a number line, we see:



We see that $f'(x)$ does not change sign at $x = -2$, so this point is not a local extremum. At $x = -1$, $f'(x)$ changes from positive to negative, so $x = -1$ is a local maximum. Finally, at $x = 3$, $f'(x)$ changes from negative to positive, so $x = 3$ is a local minimum.

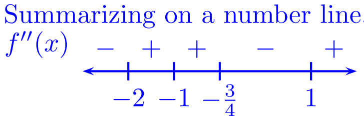
Note that we could use the Second Derivative Test as well, but it would be inconclusive at $x = -2$, so we would have to resort to the First Derivative Test to classify that critical point.

Answer: Local min(s) at $x =$ 3

Answer: Local max(es) at $x =$ -1

c. [5 points] Find the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE.
Justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: Inflection points of $f(x)$ can occur whenever $f''(x)$ is zero or undefined. In this case, $f''(x)$ is zero at $x = -2$, $-3/4$, and 1 and undefined at $x = -1$. We must check whether the sign of $f''(x)$ actually changes at each of these points.
For $x < -2$, we have $f''(x) = \frac{- \cdot - \cdot -}{+} = -$.
For $-2 < x < -1$, we have $f''(x) = \frac{+ \cdot - \cdot -}{+} = +$.
For $-1 < x < -3/4$, we have $f''(x) = \frac{+ \cdot - \cdot -}{+} = +$.
For $-3/4 < x < 1$, we have $f''(x) = \frac{+ \cdot - \cdot +}{+} = -$.
For $x > 1$, we have $f''(x) = \frac{+ \cdot + \cdot +}{+} = -$.
Summarizing on a number line, we see:



Because $f''(x)$ does not change sign at $x = -1$, this is not an inflection point. Since $f''(x)$ changes sign at $x = -2$, $-3/4$, and 1, we conclude that these are the inflection points of $f(x)$.

Answer: Inflection point(s) at $x =$ $-2, -3/4, 1$