5. [9 points] Consider the curve $\mathcal{C}$ defined by

$$e^{\pi xy} = ay^2 + x^2$$

where a is a positive constant.

a. [6 points] For this curve $\mathcal{C}$, find a formula for $\dfrac{dy}{dx}$ in terms of x and y. The constant a may appear in your answer. Remember to show every step of your work clearly.

Solution: We differentiate both sides of the equation defining $\mathcal{C}$ with respect to x and then solve for $\dfrac{dy}{dx}$.

$$\begin{aligned}
\frac{d}{dx}\left(e^{\pi xy}\right) &= \frac{d}{dx}\left(ay^2 + x^2\right) \\
e^{\pi xy}\frac{d}{dx}(\pi xy) &= 2ay\frac{dy}{dx} + 2x \\
\pi e^{\pi xy}\left(x\frac{dy}{dx} + y\right) &= 2ay\frac{dy}{dx} + 2x \\
\pi x e^{\pi xy}\frac{dy}{dx} + \pi y e^{\pi xy} &= 2ay\frac{dy}{dx} + 2x \\
\pi x e^{\pi xy}\frac{dy}{dx} - 2ay\frac{dy}{dx} &= 2x - \pi y e^{\pi xy} \\
\frac{dy}{dx}\left(\pi x e^{\pi xy} - 2ay\right) &= 2x - \pi y e^{\pi xy} \\
\frac{dy}{dx} &= \frac{2x - \pi y e^{\pi xy}}{\pi x e^{\pi xy} - 2ay}
\end{aligned}$$

Answer: $\dfrac{dy}{dx} = \dfrac{2x - \pi y e^{\pi xy}}{\pi x e^{\pi xy} - 2ay}$

b. [1 point] Let $a = 1$. Exactly one of the following points (x, y) lies on the curve $\mathcal{C}$. Circle that one point.

$(0, 3)$ $(1, 2)$ $(2, -1)$ $\boxed{(0, -1)}$ $(e^{\pi}, 0)$

Solution: When $a = 1$, the point $(0, -1)$ satisfies the equation defining the curve $\mathcal{C}$.

c. [2 points] With $a = 1$ as above, is the tangent line to the curve $\mathcal{C}$ at the point you chose in (b) increasing, decreasing, or is there not enough information to determine this? Circle your one choice and then justify your answer.

The tangent line to the curve $\mathcal{C}$ at the point circled in (b) is

i. $\boxed{\text{increasing.}}$ ii. decreasing. iii. NOT ENOUGH INFORMATION

Justification:

Solution: The slope of this tangent line is $\frac{dy}{dx}$ evaluated at $(0, -1)$, i.e. the slope of the tangent line is $\frac{2(0) - \pi(-1)e^{\pi(0)(-1)}}{\pi(0)e^{\pi(0)(-1)} - 2(1)(-1)} = \frac{\pi}{2}$. Since $\frac{\pi}{2} > 0$, the tangent line has positive slope so is increasing.