**7.** [10 points] To aid in Elphaba's escape, Walt has concocted a supplement that will make her stronger and more agile. The concentration of the supplement in Elphaba's system, in mg/ml, $t$ minutes after it is administered is given by the following formula:

$$T(t) = \begin{cases} at^3 & 0 \le t \le 5 \\ b(t-6)^2 + 10 & 5 < t \le 7 \end{cases}$$

where $a$ and $b$ are constants.

**a.** [7 points] Given that $T(t)$ is differentiable, find $a$ and $b$. Give your answers in exact form.

*Solution:* Because the two pieces of the function are polynomials, the only point at which the function could fail to be differentiable is at $t = 5$. In order to be differentiable at $t = 5$, $T(t)$ must be continuous there. So $\lim_{t\to 5^-} T(t) = \lim_{t\to 5^+} T(t)$. Now,

$$\lim_{t\to 5^-} T(t) = \lim_{t\to 5^-} at^3 = a(5^3) = 125a$$

and

$$\lim_{t\to 5^+} T(t) = \lim_{t\to 5^+} b(t-6)^2 + 10 = b(5-6)^2 + 10 = b + 10.$$

So we must have $125a = b + 10$.

In order for $T(t)$ to be differentiable at $t = 5$, the slope of the tangent line to the graph of $y = at^3$ and $t = 5$ must be the same as the slope of the tangent line to the graph of $y = b(t-6)^2 + 10$ at $t = 5$. The slope of the tangent line to the graph of $y = at^3$ at $t = 5$ is $3at^2$ evaluated at $t = 5$, which is $75a$. The slope of the tangent line to the graph of $y = b(t-6)^2 + 10$ at $t = 5$ is $2b(t-6)$ evaluated at $t = 5$, which is $-2b$. Hence $75a = -2b$.

So in order for $T(t)$ to be differentiable, we must have $125a = b + 10$ and $75a = -2b$. Solving these two equations simultaneously we find $a = \frac{4}{65}$ and $b = -\frac{30}{13}$.

**Answer:** $a =$ $\frac{4}{65}$ and $b =$ $-\frac{30}{13}$

**b.** [3 points] Using the values of $a$ and $b$ you found in part (a), give a formula for the tangent line to the graph of $y = T(t)$ at $t = 5$.

*Solution:* $T(5) = a(5^3) = \frac{4}{65}(125) = \frac{100}{13}$ and $T'(5) = a(3(5^2)) = \frac{4}{65}(75) = \frac{60}{13}$. So a formula for the tangent line to the graph of $y = T(t)$ at $t = 5$ is $y = \frac{100}{13} + \frac{60}{13}(t-5)$.

**Answer:** $y =$ $\frac{100}{13} + \frac{60}{13}(t-5)$ **or** $\frac{60}{13}t - \frac{200}{13}$