10. [9 points] Consider the function h defined by $$h(x) = \begin{cases} Ax^4 & \text{if } x < 2 \\ Bx^3 + 80\ln\left(\frac{x}{2}\right) & \text{if } x \geq 2 \end{cases}$$
where A and B are constants.

a. [6 points] Find values of A and B so that h is differentiable.
Remember to show your work clearly.

Solution: If h is differentiable, it must be continuous, so, in particular,

$$\begin{aligned} \lim_{x\to 2^-} h(x) &= \lim_{x\to 2^+} h(x) \\ A(2)^4 &= B(2)^3 + 80\ln(2/2) \\ 16A &= 8B \\ 2A &= B. \end{aligned}$$

Note that $\frac{d}{dx}\left(Ax^4\right) = 4Ax^3$ and $\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right) = 3Bx^2 + 80(\frac{2}{x})(\frac{1}{2}) = 3Bx^2 + \frac{80}{x}$. and that both Ax^4 and $Bx^3 + 80\ln\left(\frac{x}{2}\right))$ are differentiable at $x = 2$.
In order for $h(x)$ to be differentiable at $x = 2$, $h'(x)$ must exist at $x = 2$. In particular,

$$\begin{aligned} \lim_{k\to 0^-} \frac{h(2+k) - h(2)}{k} &= \lim_{k\to 0^+} \frac{h(2+k) - h(2)}{k} \\ \left(\frac{d}{dx}(Ax^4)\right)\Big|_{x=2} &= \left(\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right)\right)\Big|_{x=2} \\ (4Ax^3)\Big|_{x=2} &= \left(3Bx^2 + \frac{80}{x}\right)\Big|_{x=2} \quad \text{(i.e. derivatives of the two pieces are equal at } x = 2) \\ 32A &= 12B + 40. \end{aligned}$$

Since $B = 2A$, we therefore find that

$$\begin{aligned} 32A &= 24A + 40 \\ 8A &= 40 \\ A &= 5 \end{aligned}$$

and hence $B = 2A = 2(5) = 10$.

Answer: $A =$ 5 and $B =$ 10

b. [3 points] Using the values of A and B you found in part **a.**, find the tangent line approximation for $h(x)$ near $x = 1$.

Solution: First, notice that
$$h(1) = 5(1)^4 = 5$$
and
$$h'(1) = 4(5)(1)^3 = 20.$$
So the tangent line approximation for $h(x)$ near $x = 1$ is $y = 5 + 20(x-1) = 20x - 15$.

Answer: The tangent line approximation is given by $y =$ $5 + 20(x-1)$ **(or** $20x - 15$**)**