10. [9 points] Consider the function $h$ defined by

$$h(x) = \begin{cases} Ax^4 & \text{if } x < 2 \\ Bx^3 + 80 \ln \left( \frac{x}{2} \right) & \text{if } x \geq 2 \end{cases}$$

where $A$ and $B$ are constants.

a. [6 points] Find values of $A$ and $B$ so that $h$ is differentiable.

**Solution:** If $h$ is differentiable, it must be continuous, so, in particular,

$$\lim_{x \to 2^-} h(x) = \lim_{x \to 2^+} h(x)$$

$$A(2)^4 = B(2)^3 + 80 \ln(2/2)$$

$$16A = 8B$$

$$2A = B.$$ 

Note that $\frac{d}{dx} (Ax^4) = 4Ax^3$ and $\frac{d}{dx} (Bx^3 + 80 \ln \left( \frac{x}{2} \right)) = 3Bx^2 + 80 \left( \frac{1}{x} \right) = 3Bx^2 + \frac{80}{x}.

and that both $Ax^4$ and $Bx^3 + 80 \ln \left( \frac{x}{2} \right)$ are differentiable at $x = 2.$

In order for $h(x)$ to be differentiable at $x = 2$, $h'(x)$ must exist at $x = 2.$ In particular,

$$\lim_{k \to 0^-} \frac{h(2 + k) - h(2)}{k} = \lim_{k \to 0^+} \frac{h(2 + k) - h(2)}{k}$$

$$\left( \frac{d}{dx} (Ax^4) \right) \bigg|_{x=2} = \left( \frac{d}{dx} (Bx^3 + 80 \ln \left( \frac{x}{2} \right)) \right) \bigg|_{x=2}$$

$$(4Ax^3) \bigg|_{x=2} = \left( 3Bx^2 + \frac{80}{x} \right) \bigg|_{x=2} \quad \text{(i.e. derivatives of the two pieces are equal at } x = 2)$$

$$32A = 12B + 40.$$ 

Since $B = 2A$, we therefore find that

$$32A = 24A + 40$$

$$32A = 24A + 40$$

$$8A = 40$$

$$A = 5$$


**Answer:** $A = \boxed{5}$ and $B = \boxed{10}$

b. [3 points] Using the values of $A$ and $B$ you found in part a., find the tangent line approximation for $h(x)$ near $x = 1$.

**Solution:** First, notice that

$$h(1) = 5(1)^4 = 5$$

and

$$h'(1) = 4(5)(1)^3 = 20.$$ 

So the tangent line approximation for $h(x)$ near $x = 1$ is $y = 5 + 20(x - 1) = 20x - 15.$

**Answer:** The tangent line approximation is given by $y = \frac{5 + 20(x - 1)}{20x - 15} \text{ (or } 20x - 15)$