5. [15 points] Suppose $g(x)$ is a differentiable function defined for all real numbers x.
The derivative and second derivative of $g(x)$ are given by

$$g'(x) = x^2(x+4)(x+2)^{1/3} \qquad \text{and} \qquad g''(x) = \frac{2x(x+3)(5x+8)}{3(x+2)^{2/3}}.$$

a. [2 points] Find the x-coordinates of all critical points of $g(x)$.
If there are none, write "NONE".

Answer: Critical point(s) of $g(x)$ at $x =$ $-4, -2, 0$

b. [2 points] Find the x-coordinates of all critical points of $g'(x)$.
If there are none, write "NONE".

Answer: Critical point(s) of $g'(x)$ at $x =$ $-3, -2, -\frac{8}{5}, 0$

c. [6 points] Find the x-coordinates of all local maxima and local minima of $g(x)$.
If there are none of a particular type, write "NONE". Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema.

Solution: We know from part **a.** that the critical points of $g(x)$ are $-4, -2, 0$. Notice that the Second Derivative Test is inconclusive, because $g''(-2)$ does not exist and $g''(0) = 0$. So we must use the First Derivative Test. Notice that the factor x^2 is always non-negative, $(x+4)$ is negative for $x < -4$ and positive for $x > -4$, and $(x+2)^{1/3}$ is negative for $x < -2$ and positive for $x > -2$. This gives us the resulting signs:

Interval	$x < -4$	$-4 < x < -2$	$-2 < x < 0$	$x > 0$
Sign of $g'(x)$	$+ \cdot - \cdot - = +$	$+ \cdot + \cdot - = -$	$+ \cdot + \cdot + = +$	$+ \cdot + \cdot + = +$

So $g(x)$ has a local maximum at $x = -4$ and a local minimum at $x = -2$. There is neither a local maximum nor a local minimum at $x = 0$.

Answer: Local max(es) at $x =$ -4 and Local min(s) at $x =$ -2

d. [5 points] Find the x-coordinates of all inflection points of $f(x)$.
If there are none, write "NONE". Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all inflection points.

Solution: We know from part **b.** that $g''(x)$ is 0 or undefined (i.e., $g'(x)$ has a critical point or is undefined) at $x = -3, -2, -\frac{8}{5}, 0$. To determine whether these are actually inflection points (where concavity changes), we must test the sign of the second derivative on either side of each point. Notice that the factor $2x$ is negative for $x < 0$ and positive for $x > 0$, the factor $(x+3)$ is negative for $x < -3$ and positive for $x > -3$, the factor $(5x+8)$ is negative for $x < -\frac{8}{5}$ and positive for $x > -\frac{8}{5}$, and the factor $(x+2)^{-2/3}$ is always non-negative. We find the following signs for $g''(x)$:

Interval	$x < -3$	$-3 < x < -2$	$-2 < x < -\frac{8}{5}$	$-\frac{8}{5} < x < 0$	$x > 0$
Sign of $g''(x)$	$\frac{- \cdot - \cdot -}{+} = -$	$\frac{- \cdot + \cdot -}{+} = +$	$\frac{- \cdot + \cdot -}{+} = +$	$\frac{- \cdot + \cdot +}{+} = -$	$\frac{+ \cdot + \cdot +}{+} = +$

So $g(x)$ has inflection points at $x = -3$, $x = -\frac{8}{5}$, and $x = 0$ but it does not have an inflection point at $x = -2$.

Answer: Inflection point(s) at $x =$ $-3, -\frac{8}{5}, 0$