5. [12 points] Let

$$f(x) = x(x-4)^{4/5}e^{-x} \qquad \text{and} \qquad f'(x) = \frac{(5-x)(5x-4)e^{-x}}{5\sqrt[5]{x-4}}.$$

Note that the domain of $f(x)$ is $(-\infty, \infty)$.

a. [6 points] Find all values of x at which $f(x)$ has a local extremum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all local extrema. For each answer blank below, write NONE if appropriate.

Solution: First we look for:

- Values of x for which $f'(x) = 0$:

$$\begin{aligned} \frac{(5-x)(5x-4)e^{-x}}{5\sqrt[5]{x-4}} &= 0 \\ (5-x)(5x-4)e^{-x} &= 0 \\ 5-x = 0 \quad &\text{or} \quad 5x-4=0 \\ x = 5 \quad &\text{or} \quad x = 0.8. \end{aligned}$$

  Notice that $e^{-x} \neq 0$ for all values o x. Hence the only solutions are $x = 0.8$ and $x = 5$.
- Values of x for which $f'(x)$ is undefined: The function $f'(x) = \frac{(5-x)(5x-4)e^{-x}}{5\sqrt[5]{x-4}}$ is undefined when.

$$\begin{aligned} 5\sqrt[5]{x-4} &= 0 \\ \sqrt[5]{x-4} &= 0 \\ x-4 &= 0 \quad \text{which yields} \quad x = 4. \end{aligned}$$

  Finally we notice that all of these points are in the domain of $f(x)$, hence the critical points of $f(x)$ are $x = 0.8, 4, 5$.

We use the first derivative test to classify all the critical points of $f(x)$ by either finding the signs of $f'(x)$ around the critical points or computing its values:

$$\begin{aligned} f'(0) &= \frac{(+)(-)(+)}{-} = + \quad &\text{or} \quad f'(0) \approx 3.031 \\ f'(1) &= \frac{(+)(+)(+)}{-} = - \quad &\text{or} \quad f'(1) \approx -0.236 \\ f'(4.5) &= \frac{(+)(+)(+)}{+} = + \quad &\text{or} \quad f'(4.5) \approx 0.023 \\ f'(6) &= \frac{(-)(+)(+)}{+} = - \quad &\text{or} \quad f'(6) \approx -0.011 \end{aligned}$$

Hence $x = 0.8, 5$ are local maximums and $x = 4$ is a local minimum.

Answer:
Local max(es) at $x =$ $0.8, 5$ Local min(s) at $x =$ 4

b. [6 points] Find the values of x for which $f(x)$ attains a global maximum and global minimum. Use calculus to find and justify your answers, and be sure to show enough evidence to demonstrate that you have found all global extrema. Write NONE if appropriate.

Solution:

x	0.8	4	5
$f(x)$	0.911	0	0.033

and

$$\lim_{x\to\infty} x(x-4)^{4/5}e^{-x} = \lim_{x\to\infty} \frac{x(x^{4/5})}{e^x} = \lim_{x\to\infty} \frac{x^{9/5}}{e^x} = 0$$

$$\lim_{x\to-\infty} x(x-4)^{4/5}e^{-x} = \lim_{x\to-\infty} x^{9/5}e^{-x} = -\infty.$$

The last limit follows from the fact that $\lim\limits_{x\to-\infty} x^{9/5} = -\infty$ and $\lim\limits_{x\to-\infty} e^{-x} = \infty$. Hence the function $f(x)$ has a global maximum at $x = 0.8$ and no global minimum.