9. [13 points] Let $\mathcal{C}$ be the curve defined by the equation

$$\ln(xy) = x^2.$$

Note that the curve $\mathcal{C}$ satisfies

$$\frac{dy}{dx} = \frac{y(2x^2-1)}{x}.$$

a. [4 points] Exactly one of the following points lies on $\mathcal{C}$. Circle that one point.

$(0,1)$ $(1,0)$ $(1,1)$ $\boxed{(1,e)}$ $(e,1)$ (e,e)

Then find an equation for the line tangent to $\mathcal{C}$ at the point you chose above.

Solution: Plugging $(x,y) = (1,e)$ into $\ln(xy) = \ln(1(e)) = 1$ and $x^2 = 1^2 = 1$.Hence the point $(1,e)$ is in the graph of the equation $\ln(xy) = x^2$.
The slope m of the tangent line at $(1,e)$ is

$$m = \frac{dy}{dx}\Big|_{(1,e)} = \frac{e(2(1)^2-1)}{1} = e.$$

Hence the equation of the tangent line is given by $y = e + e(x-1)$.

b. [4 points] Find all points on $\mathcal{C}$ with a horizontal tangent line. Give your answers as ordered pairs (coordinates). Show your work. Write NONE if no such points exist.

Solution: We know $\mathcal{C}$ has a horizontal tangent line when $\frac{dy}{dx} = 0$. This happens when $y(2x^2-1) = 0$, which happens when $y = 0$ or $2x^2 - 1 = 0$. The latter is true when $x = \pm\frac{1}{\sqrt{2}}$.
Now we plug these values into the equation for $\mathcal{C}$ to find the other coordinates.

1. When $y = 0$, the equation for the x-coordinate is $\ln(x(0)) = 0^2$. This equation has no solutions then there is no point with horizontal tangent lines of the form $(x,0)$.
2. When $x = \frac{1}{\sqrt{2}}$, then

$$\ln\left(\frac{1}{\sqrt{2}}y\right) = \left(\frac{1}{\sqrt{2}}\right)^2 \quad \text{or} \quad \ln\left(\frac{1}{\sqrt{2}}y\right) = 0.5$$
$$\frac{1}{\sqrt{2}}y = e^{0.5} \quad \text{and} \quad y = \sqrt{2}\sqrt{e} = \sqrt{2e}$$

3. Similarly when $x = -\frac{1}{\sqrt{2}}$

$$\ln\left(-\frac{1}{\sqrt{2}}y\right) = \left(-\frac{1}{\sqrt{2}}\right)^2 \quad \text{yields} \quad y = -\sqrt{2e}$$

Answer: $(x,y) =$ $\left(\frac{1}{\sqrt{2}}, \sqrt{2e}\right), \left(-\frac{1}{\sqrt{2}}, -\sqrt{2e}\right)$

c. [5 points] Consider the curve $\mathcal{D}$ defined by

$$y + 2^x y^4 = 3 - \sin(x^2).$$

Find a formula for $\dfrac{dy}{dx}$ in terms of x and y. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution:

$$\begin{aligned}
\frac{dy}{dx} + \ln 2 \cdot 2^x y^4 + 2^x \cdot 4y^3 \frac{dy}{dx} &= -2x\cos(x^2) \\
\frac{dy}{dx} + 2^x \cdot 4y^3 \frac{dy}{dx} &= -2x\cos(x^2) - \ln 2 \cdot 2^x y^4 \\
\frac{dy}{dx}\left(1 + 2^x \cdot 4y^3\right) &= -2x\cos(x^2) - \ln 2 \cdot 2^x y^4 \\
\frac{dy}{dx} &= \frac{-2x\cos(x^2) - \ln 2 \cdot 2^x y^4}{1 + 2^x \cdot 4y^3}
\end{aligned}$$