3. [12 points] The Public Transit Authorities (PTA) are designing rain shelters for their bus stops. They decide to place a roof in the shape of half a cylinder on four vertical legs of height $y$ feet. The four legs are placed in a rectangle on the ground with width $x$ feet and length $y$ feet.

The costs of production are:

- $25 for each foot of the total length of the legs,
- $40 for each square foot of the area of the roof.

The following formulas may be useful in this problem:

- the surface area of a cylinder of radius $r$ and length $\ell$ is $2\pi r \ell$,
- the volume of a cylinder of radius $r$ and length $\ell$ is $\pi r^2 \ell$.

The PTA would like to spend exactly $5000 on one rain shelter.

**a.** [5 points] Find a formula for $y$ in terms of $x$.

**Solution:** We have that

$$25 \cdot (4y) + 40 \cdot \frac{1}{2} \left(2\pi \left(\frac{x}{2}\right)\right) y = 5000$$

so

$$y(100 + 20\pi x) = 5000,$$

$$y = \frac{5000}{100 + 20\pi x}.$$  

**Answer:** $y = \frac{250}{5 + \pi x}$.

**b.** [4 points] Find a formula for the total volume in cubic feet covered by the shelter, $V(x)$, if the width of the dashed rectangle has length $x$ feet.

**Solution:** The volume is

$$V = xy^2 + \frac{1}{2} \pi \left(\frac{x}{2}\right)^2 y,$$

and hence

**Answer:** $V(x) = x \cdot \left(\frac{250}{5 + \pi x}\right)^2 + \frac{1}{2} \pi \left(\frac{x}{2}\right)^2 \cdot \left(\frac{250}{5 + \pi x}\right)$.

**c.** [3 points] The PTA wants to make sure that each of the sides of the rectangle has length at least 5 feet, and the height (that is, $y$) of the shelter is at least 8 feet. In the context of the problem, what is the domain of the function $V(x)$?

**Solution:** We know that $x \geq 5$ and also $y \geq 8$. Therefore, $\frac{250}{5 + \pi x} = y \geq 8$  

$$40 + 8\pi x \quad x \leq \frac{210}{8\pi} = \frac{105}{4\pi}.$$  

**Answer:** $5 \leq x \leq \frac{105}{4\pi}$.