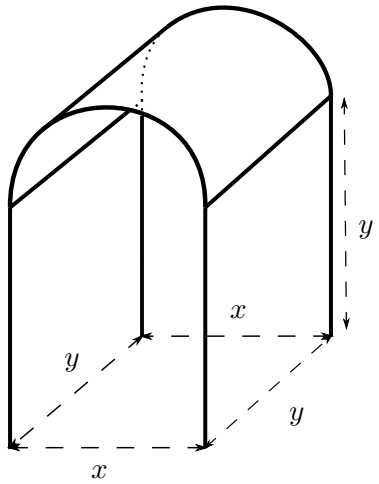
3. [12 points] The Public Transit Authorities (PTA) are designing rain shelters for their bus stops. They decide to place a roof in the shape of half a cylinder on four vertical legs of height y feet. The four legs are placed in a *rectangle* on the ground with width x feet and length y feet.

The costs of production are:

- \$25 for each foot of the total length of the legs,
- \$40 for each square foot of the area of the roof.

The following formulas may be useful in this problem:

- the surface area of a cylinder of radius r and length ℓ is $2\pi r\ell$,
- the volume of a cylinder of radius r and length ℓ is $\pi r^2\ell$.



The PTA would like to spend exactly \$5000 on one rain shelter.

a. [5 points] Find a formula for y in terms of x.

Solution: We have that

$$25 \cdot (4y) + 40 \cdot \frac{1}{2}\left(2\pi\left(\frac{x}{2}\right)\right) y = 5000$$

so

$$y(100 + 20\pi x) = 5000,$$

$$y = \frac{5000}{100 + 20\pi x}.$$

Answer: $y = \dfrac{250}{5 + \pi x}$.

b. [4 points] Find a formula for the total volume in cubic feet covered by the shelter, $V(x)$, if the width of the dashed rectangle has length x feet.

Solution: The volume is

$$V = xy^2 + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 y,$$

and hence **Answer:** $V(x) = x \cdot \left(\frac{250}{5+\pi x}\right)^2 + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \cdot \left(\frac{250}{5+\pi x}\right)$.

c. [3 points] The PTA wants to make sure that *each* of the sides of the rectangle has length at least 5 feet, and the height (that is, y) of the shelter is at least 8 feet. In the context of the problem, what is the domain of the function $V(x)$?

Solution: We know that $x \geq 5$ and also $y \geq 8$. Therefore, $\dfrac{250}{5 + \pi x} = y \geq 8$ $250 \geq 40 + 8\pi x$ $x \leq \dfrac{210}{8\pi} = \dfrac{105}{4\pi}$.

Answer: $5 \leq x \leq \dfrac{105}{4\pi}$