5. [8 points] Consider the function $h(x)$ where k and A are constants:

$$h(x) = \begin{cases} 2x+1 & x \leq k \\ (x-A)^2+2 & x > k \end{cases}$$

a. [5 points] There is exactly one choice of the constants A and k that make $h(x)$ differentiable. Find these values of A and k.

Solution: Since both pieces of the function are differentiable, the only place where $h(x)$ might not be differentiable is at the point $x = k$. To be differentiable at $x = k$, the function must first be continuous at $x = k$, so we set the two parts of the piecewise function equal to each other at $x = k$:

$$2k + 1 = (k - A)^2 + 2.$$

We also need to set their derivatives equal to each other at $x = k$:

$$\begin{aligned} 2 &= 2(k-A) \\ 2 &= 2k - 2A \\ 2 + 2A &= 2k \\ 1 + A &= k. \end{aligned}$$

Substituting this into the first equation we get

$$\begin{aligned} 2(1+A)+1 &= ((1+A)-A)^2+2 \\ 2+2A+1 &= (1)^2+2 \\ 2A+3 &= 3 \\ 2A &= 0 \\ A &= 0, \end{aligned}$$

and since $k = 1 + A$ we have that $k = 1$.

Answer: $A =$ 0 **Answer:** $k =$ 1

b. [3 points] If $A > k$, then $h(x)$ has two critical points. What are the x-coordinates of these points? Your answers may be in terms of A and/or k. Show work or briefly explain your reasoning.

Solution: If $A > k$, then we certainly cannot be in the situation where $A = 0$ and $k = 1$ from part a. Thus $h(x)$ will not be differentiable at k and $x = k$ is one of the critical points.

Since the derivative of $2x + 1$ is 2, there are no critical points less than k.

Finally, the derivative of $(x - A)^2 + 2$ is $2(x - A)$, which is zero when $x = A$. Since $A > k$, this point does fall in the domain of the second piece, and so $x = A$ is the second critical point.

Answer: Critical point(s) at $x =$ k **and** A

