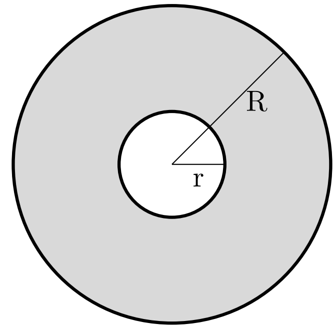
**8.** [8 points]

Kristen is machining a metal washer to fix her broken down motorcycle. A washer is a flat, circular piece of metal with a hole in the middle. Kristen's washer is depicted by the shaded region in the figure to the right. The washer has an inner radius of $r$ centimeters and an outer radius of $R$ centimeters. The area of the washer must be exactly 5 square centimeters, and $r$ must be at least 1 centimeter.




**a.** [3 points] Find a formula for $r$ in terms of $R$.

*Solution:* The area of the washer is the difference between the outer circle's area and inner circle's area. So, since this must be 5 square centimeters we have $\pi R^2 - \pi r^2 = 5$, so $r^2 = \dfrac{\pi R^2 - 5}{\pi}$, and $r = \sqrt{\dfrac{\pi R^2 - 5}{\pi}}$.

**Answer:** $r = \sqrt{\dfrac{\pi R^2 - 5}{\pi}}$

**b.** [2 points] The structural integrity of the washer depends on both its inner radius and its outer radius. Specifically, the structural integrity is given by the equation

$$S = 32R\left(\ln(rR + 1) + 7\right).$$

Express $S$ as a function of $R$. *Your answer should not include $r$.*

*Solution:* We substitute our answer from part a. into the formula for $S$.

**Answer:** $S(R) = 32R(\ln\left(R\sqrt{\dfrac{\pi R^2 - 5}{\pi}} + 1\right) + 7)$

**c.** [3 points] What is the domain of $S(R)$ in the context of this problem? You may give your answer as an interval or using inequalities.

*Solution:* We are told that $r$ must be at least 1. When $r = 1$, we have

$$\pi R^2 - \pi r^2 = 5$$
$$\pi R^2 - \pi = 5$$
$$R^2 = \frac{5 + \pi}{\pi}$$
$$R = \sqrt{\frac{5 + \pi}{\pi}}.$$

This is the smallest possible value of $R$, because if we make $r$ larger, $R$ must also be made larger so that the area of the washer can remain 5 square centimeters. There is no upper bound on how large $R$ can be.

**Answer:** $[\sqrt{\dfrac{5 + \pi}{\pi}}, \infty)$