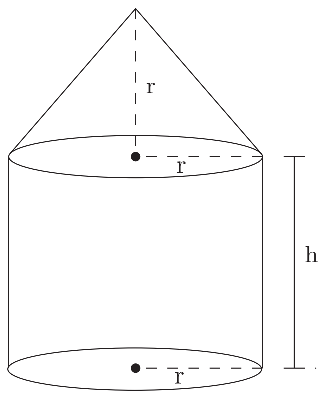
Note: exam problem numbering is off by 1

8. [9 points]



A city is in the planning stages of building a shed to store road salt. One design being considered is shown above. The sides would be a cylinder of radius r feet and height h feet, and the roof would be a cone in which both the radius and height are equal to r feet. (The city does not need to build a floor.) The cost of the materials for this shed, in dollars, is

$$4\pi r^2 + 4\pi rh.$$

If the city wants to spend \$20,000 on materials, what values of r and h will maximize the volume of the shed? Give your answers to at least two decimal places, and be sure to find and justify your answers using calculus.

Note that the volume of a cone with radius R and height H is $\frac{1}{3}\pi R^2 H$.

Solution: The total amount, in dollars, the city spends on a shed of height h and radius r is given by

$$20,000 = 4\pi r^2 + 4\pi rh.$$

Solving for h, we obtain

$$h = \frac{20,000 - 4\pi r^2}{4\pi r}.$$

Noting that the height of the cone is the same as its height, we have that the total volume of the shed is given by

$$V = \pi r^2 h + \frac{1}{3}\pi r^3.$$

Substituting our formula for h, we obtain

$$V = \pi r^2 \frac{20,000 - 4\pi r^2}{4\pi r} + \frac{1}{3}\pi r^3.$$

This then simplifies to

$$V = 5000r - \pi r^3 + \frac{1}{3}\pi r^3.$$

Taking the derivative, we obtain

$$V' = 5000 - 3\pi r^2 + \pi r^2 = 5000 - 2\pi r^2.$$

This is defined everywhere, and we find that the derivative vanishes at $\pm\sqrt{2500/\pi}$. We know that r cannot be negative, and when $r = \sqrt{2500/\pi}$, we have $h = \dfrac{10,000}{4\pi\sqrt{2500/\pi}} = \sqrt{2500/\pi}$, both of which are positive, meaning the critical point is in our domain.

To show that volume is maximized, we only need to show that our lone critical point is the location of a local max (since it is the only critical point). To do this, we use the second derivative test. We have

$$V'' = -4\pi r$$

, which is negative at $r = \sqrt{2500/\pi}$, meaning that by the second derivative test we do in fact have a local max.

Thus the volume is maximized when $r = \sqrt{2500/\pi} \approx 28.21$ feet and $h = \dfrac{10,000}{4\pi\sqrt{2500/\pi}} \approx 28.21.$