5. [10 points] The graph of the function $f(x)$ is shown below. Note that $f(x)$ is linear on the interval $(-3, 1)$.

![Graph of $f(x)$]

a. [6 points] The function $g(x)$ is given by the equation

$$g(x) = \begin{cases} 
  e^{px} & x \leq 0 \\
  Cf(x) & x > 0 
\end{cases}$$

where $C$ and $p$ are constants and $f$ is as above. Find one pair of exact values for $C$ and $p$ such that $g(x)$ is differentiable, or write none if there are none. Be sure your work is clear.

**Solution:** We first must ensure that $g(x)$ is continuous at $x = 0$. We see that $\lim_{x \to 0^-} g(x) = e^{p \cdot 0}$ and $\lim_{x \to 0^+} g(x) = Cf(0)$, and so we solve:

$$e^{p \cdot 0} = Cf(0)$$

$$1 = C \cdot 2$$

$$\frac{1}{2} = C.$$  

We must next ensure that $\lim_{x \to 0^-} g'(x) = \lim_{x \to 0^+} g'(x)$. We see that $\lim_{x \to 0^-} g'(x) = pe^{p \cdot 0}$ and $\lim_{x \to 0^+} g'(x) = Cf'(0)$. Since $f(x)$ is linear on $(-3, 1)$, we compute its slope on the interval $[-3, 0]$ to find that $f'(0) = (2 - 1)/(0 - (-3)) = \frac{1}{3}$. We now solve

$$pe^{p \cdot 0} = Cf'(0)$$

$$p \cdot 1 = \frac{1}{2} \cdot \frac{1}{3}$$

$$p = \frac{1}{6}.$$  

We conclude that $C = \frac{1}{2}$, $p = \frac{1}{6}$. 
Part of the graph of the function \( h(x) \) is shown below.

Note that \( h(4) = -\frac{\sqrt{2}}{3} \).

b. [2 points] Complete the following sentence.

*Because the function \( h(x) \) satisfies the hypotheses of the mean value theorem on the interval \([2, 4]\), there must be some point \( c \) with \( 2 \leq c \leq 4 \) such that...*

**Solution:** \( h'(c) = \frac{h(4)-h(2)}{4-2} = -\frac{\sqrt{2}}{6} \).

c. [2 points] On which of the following intervals does \( h(x) \) satisfy the hypotheses of the mean value theorem? List all correct answers, or write NONE.

\[
\begin{array}{ccc}
[-1,0] & [0,3] & [1,4] \\
\end{array}
\]

**Solution:** The hypotheses of the mean value theorem for a function \( h(x) \) on an interval \([a,b]\) are:

- Our function \( h(x) \) is continuous on the closed interval \([a,b]\).
- Our function \( h(x) \) is differentiable on the open interval \((a,b)\).

- These are true for \( h(x) \) on \([-1,0]\) because \( h(x) \) is constant on this interval.
- These are not true for \( h(x) \) on \([0,3]\) because we see from the graph that \( h(x) \) is not differentiable at \( x = 1 \), which is inside the open interval \((0,3)\).
- These are true for \( h(x) \) on \([1,4]\), because we see from the graph that \( h(x) \) is continuous on this interval, and the only point at which \( h(x) \) is not differentiable is \( x = 1 \), which is not inside the open interval \((1,4)\).
- We conclude that \( h(x) \) satisfies the hypotheses of the mean value theorem on \([-1,0] \text{ and } [1,4] \).