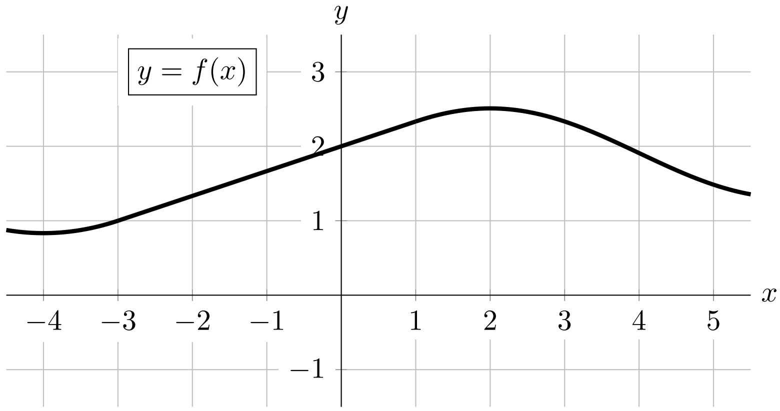
5. [10 points] The graph of the function $f(x)$ is shown below.
Note that $f(x)$ is linear on the interval $(-3, 1)$.



a. [6 points] The function $g(x)$ is given by the equation

$$g(x) = \begin{cases} e^{px} & x \leq 0 \\ Cf(x) & x > 0 \end{cases}$$

where C and p are constants and f is as above. Find one pair of **exact** values for C and p such that $g(x)$ is differentiable, or write NONE if there are none. Be sure your work is clear.

Solution: We first must ensure that $g(x)$ is continuous at $x = 0$. We see that $\lim_{x\to 0^-} g(x) = e^{p\cdot 0}$ and $\lim_{x\to 0^+} g(x) = Cf(0)$, and so we solve:

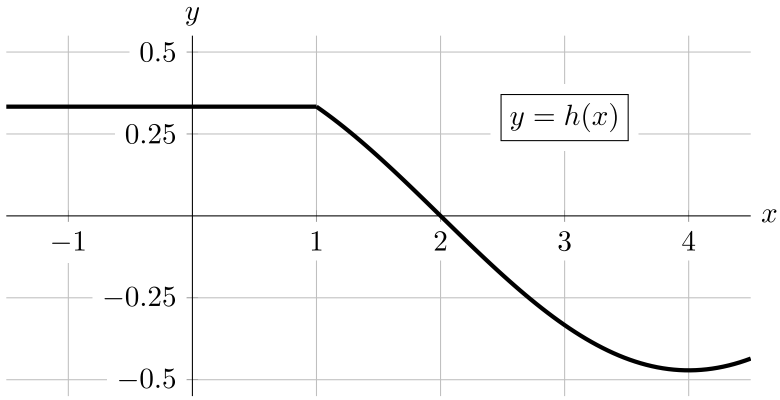
$$\begin{aligned} e^{p\cdot 0} &= Cf(0) \\ 1 &= C \cdot 2 \\ \frac{1}{2} &= C. \end{aligned}$$

We must next ensure that $\lim_{x\to 0^-} g'(x) = \lim_{x\to 0^+} g'(x)$. We see that $\lim_{x\to 0^-} g'(x) = pe^{p\cdot 0}$ and $\lim_{x\to 0^+} g'(x) = Cf'(0)$. Since $f(x)$ is linear on $(-3, 1)$, we compute its slope on the interval $[-3, 0]$ to find that $f'(0) = (2-1)/(0-(-3)) = \frac{1}{3}$. We now solve

$$\begin{aligned} pe^{p\cdot 0} &= Cf'(0) \\ p \cdot 1 &= \frac{1}{2} \cdot \frac{1}{3} \\ p &= \frac{1}{6}. \end{aligned}$$

We conclude that $\boxed{C = \frac{1}{2},\ p = \frac{1}{6}}$.

Part of the graph of the function $h(x)$ is shown below.



Note that $h(4) = -\frac{\sqrt{2}}{3}$.

b. [2 points] Complete the following sentence.

Because the function $h(x)$ *satisfies the hypotheses of the mean value theorem on the interval* $[2,4]$, *there must be some point* c *with* $2 \le c \le 4$ *such that...*

Solution: ... $h'(c) = \frac{h(4)-h(2)}{4-2} = -\frac{\sqrt{2}}{6}$.

c. [2 points] On which of the following intervals does $h(x)$ satisfy the hypotheses of the mean value theorem? List all correct answers, or write NONE.

$[-1,0]$ $[0,3]$ $[1,4]$

Solution: The hypotheses of the mean value theorem for a function $h(x)$ on an interval $[a,b]$ are:

- Our function $h(x)$ is continuous on the closed interval $[a,b]$.
- Our function $h(x)$ is differentiable on the open interval (a,b).

-These are true for $h(x)$ on $[-1,0]$ because $h(x)$ is constant on this interval.
-These are not true for $h(x)$ on $[0,3]$ because we see from the graph that $h(x)$ is not differentiable at $x=1$, which is inside the open interval $(0,3)$.
-These are true for $h(x)$ on $[1,4]$, because we see from the graph that $h(x)$ is continuous on this interval, and the only point at which $h(x)$ is not differentiable is $x=1$, which is *not* inside the open interval $(1,4)$.
-We conclude that $h(x)$ satisfies the hypotheses of the mean value theorem on $\boxed{[-1,0] \text{ and } [1,4].}$