6. [10 points] A manufacturer is constructing a closed hollow cylindrical tank out of a metal that costs \$2 per square foot. (Note that the tank has both a bottom and a top made of this same metal.) The tank's top must also be coated with a chemical that costs \$5 per square foot. The manufacturer will spend exactly \$180 on the tank.

- Find the height and radius of the cylindrical tank, in feet, so that the tank has the maximum possible volume.
- What is the maximum volume in this case, in cubic feet?

In your solution, make sure to carefully define any variables and functions you use, use calculus to justify your answers, and show enough evidence that the values you find do in fact maximize the volume.

Please note that the explanation below is much more extensive than what was required to earn full credit on the problem.

Solution: Let h denote the height of the tank, in feet, and let r denote the radius of the tank, in feet. The total cost of the tank is:

$$\text{Cost of tank} = \text{Cost of bottom} + \text{Cost of side} + \text{Cost of top}.$$

- The bottom of the tank has surface area πr^2 square feet, and costs \$2 per square foot. Therefore Cost of bottom $= 2\pi r^2$ dollars.
- The side of the tank has surface area $2\pi rh$ square feet, and costs \$2 per square foot. Therefore Cost of side $= 4\pi rh$ dollars.
- The top of our tank has surface area πr^2, and costs $\$2 + \$5 = \$7$ per square foot. Therefore Cost of top $= 7\pi r^2$ dollars.

Since we are told the manufacturer will spend \$180 on the tank, we have

$$\begin{aligned} \text{Cost of tank} &= 2\pi r^2 + 4\pi rh + 7\pi r^2 \\ 180 &= 9\pi r^2 + 4\pi rh. \end{aligned}$$

We can use this equation to solve for h in terms of r:

$$\begin{aligned} 9\pi r^2 + 4\pi rh &= 180 \\ h &= \frac{180 - 9\pi r^2}{4\pi r} = \frac{45}{\pi}r^{-1} - \frac{9}{4}r. \end{aligned}$$

We want to maximize the volume of the cylinder, so let $V(r)$ denote the volume of the cylinder, in cubic feet, as a function of r. Then

$$\begin{aligned} V(r) &= \pi r^2 h \\ &= \pi r^2\left(\frac{45}{\pi}r^{-1} - \frac{9}{4}r\right) \\ &= 45r - \frac{9\pi}{4}r^3. \end{aligned}$$

The domain of the function $V(r)$ is constrained by the following two facts:

- r cannot be less than 0.
- h cannote be less than 0.

From the first fact, we see that the domain of $V(r)$ must begin with "[0."

Solution: *(Continued)*
Let us express the second fact as a statement about r. Using the formula $h = \frac{45}{\pi}r^{-1} - \frac{9}{4}r$, we see that the second fact can be expressed as:

$$\begin{aligned} 0 &\le \frac{45}{\pi}r^{-1} - \frac{9}{4}r \\ \frac{9}{4}r &\le \frac{45}{\pi}r^{-1} \\ \frac{9}{4}r^2 &\le \frac{45}{\pi} \\ -\sqrt{\frac{60}{3\pi}} \le r &\le \sqrt{\frac{60}{3\pi}}. \end{aligned}$$

Since we already know r cannot be less than 0, this second inequality tells us that the domain of $V(r)$ must end with "$\sqrt{\frac{60}{3\pi}}\,\Big]$." Putting these together, we conclude that the domain of $V(r)$ is $\left[0, \sqrt{\frac{60}{3\pi}}\right]$. We differentiate:

$$\begin{aligned} V(r) &= 45r - \frac{9\pi}{4}r^3 \\ V'(r) &= 45 - \frac{27\pi}{4}r^2. \end{aligned}$$

We find the r-values that make this derivative equal to 0:

$$\begin{aligned} 0 &= 45 - \frac{27\pi}{4}r^2 \\ r^2 &= \frac{45 \cdot 4}{27\pi} \\ r &= \pm\sqrt{\frac{20}{3\pi}}. \end{aligned}$$

Only one of these values, $\sqrt{\frac{20}{3\pi}}$, is inside the domain $\left[0, \sqrt{\frac{60}{3\pi}}\right]$.
To use the Second Derivative Test, we compute the second derivative

$$V''(r) = -\frac{27\pi}{2}r,$$

which is negative when r is positive; in particular $V''(r) < 0$ when $r = \sqrt{\frac{20}{3\pi}}$. Therefore $V(r)$ has a local maximum at $r = \sqrt{\frac{20}{3\pi}}$. Since this is the *only* critical point of $V(r)$ on the interval $\left[0, \sqrt{\frac{60}{3\pi}}\right]$, we conclude that $V(r)$ has a *global* maximum at $r = \sqrt{\frac{20}{3\pi}}$ on this interval. Therefore the maximum volume of the tank occurs when

$$\boxed{r = \sqrt{\frac{20}{3\pi}} \text{ feet,}} \quad \boxed{h = \frac{180 - 9\pi \cdot \frac{20}{3\pi}}{4\pi\sqrt{\frac{20}{3\pi}}} \text{ feet.}}$$

The maximum volume in this case is

$$V\left(\sqrt{\frac{20}{3\pi}}\right) = \boxed{45\sqrt{\frac{20}{3\pi}} - \frac{9\pi}{4}\sqrt{\frac{20}{3\pi}}^{\,3} \text{ cubic feet.}}$$