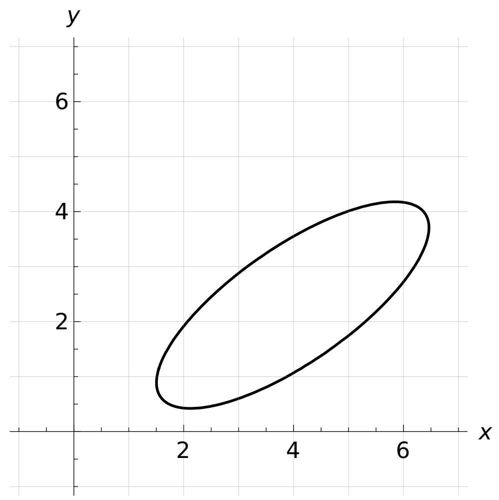
7. [4 points] A curve $\mathcal{C}$ gives y as an implicit function of x and satisfies

$$\frac{dy}{dx} = \frac{2xy}{3y^2 - x^2} \quad \text{which can be factored and rewritten as} \quad \frac{dy}{dx} = \frac{2xy}{(\sqrt{3}y - x)(\sqrt{3}y + x)}.$$

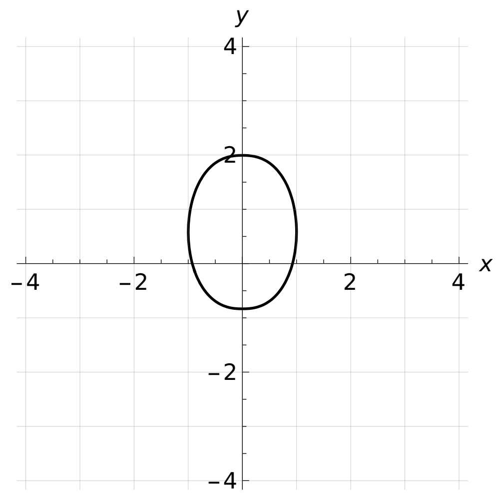
One of the following graphs is the graph of the curve $\mathcal{C}$. Write the letter corresponding to that graph.

Hint: Look for horizontal and vertical tangent lines.

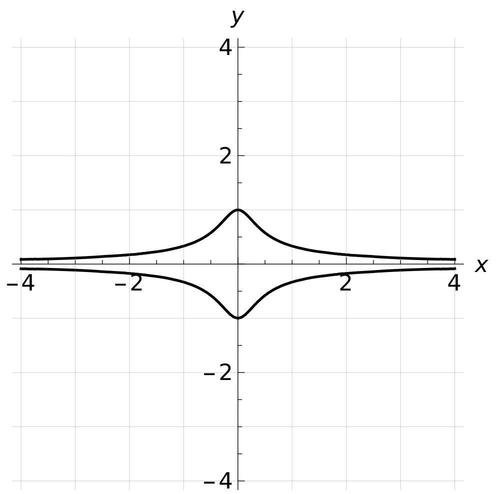
A.



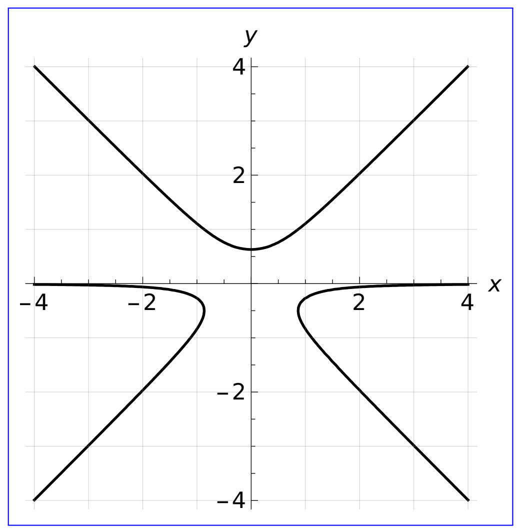
B.



C.



D.



Solution: The formula $\frac{dy}{dx} = \frac{2xy}{(\sqrt{3}y - x)(\sqrt{3}y + x)}$ tells us that $\mathcal{C}$ has a horizontal tangent line only when $x = 0$ or $y = 0$, and $\mathcal{C}$ has a vertical tangent line only when $x = \sqrt{3}y$ or $x = -\sqrt{3}y$.

- Option A has horizontal tangent lines when neither $x = 0$ nor $y = 0$, and so cannot be the graph of $\mathcal{C}$.
- Option B has tangent lines that are not horizontal when $y = 0$, and so cannot be the graph of $\mathcal{C}$.
- The equations $x = \sqrt{3}y$ and $x = -\sqrt{3}y$ are equations for lines of slope $1/\sqrt{3}$ and $-1/\sqrt{3}$ passing through the origin. The graph of $\mathcal{C}$ therefore must have a vertical tangent line whenever it intersects one of these lines. Option C passes through the y-axis, and also has the x-axis as a horizontal asymptote. Therefore Option C must pass through every non-horizontal line, and in particular, the two lines we have just described. But Option C has no vertical tangent lines, and so we conclude it cannot be the graph of $\mathcal{C}$.
- We have eliminated every other option, and so we conclude that Option D is the graph of $\mathcal{C}$.