8. [11 points] Suppose $J(x)$ is a continuous function defined for all real numbers x. The **derivative** and **second derivative** of $J(x)$ are given by

$$J'(x) = \frac{x^2(x-1)}{\sqrt[3]{x+4}} \qquad \text{and} \qquad J''(x) = \frac{x(8x^2+31x-24)}{3\left(\sqrt[3]{x+4}\right)^4}.$$

a. [2 points] Find the x-coordinates of all critical points of $J(x)$. If there are none, write NONE.

Solution: The critical points of $J(x)$ occur when $J'(x) = 0$ or $J'(x)$ does not exist. We see that $J'(x) = 0$ when $x = 0$ and $x = 1$, and we see that $J'(x)$ does not exist when $x = -4$. We conclude that the critical points of $J(x)$ occur at $\boxed{x = 0, 1, \text{ and } -4}$.

Throughout parts **b.** and **c.** below, you must use calculus to find and justify your answers. Make sure your final conclusions are clear, and that you show enough evidence to justify those conclusions.

b. [5 points] Find the x-coordinates of

i. all local minima of $J(x)$ and

ii. all local maxima of $J(x)$.

If there are none of a particular type, write NONE.

Solution: We use the First Derivative Test. Observe that x^2 is always positive unless $x = 0$.

- On the interval $(-\infty, -4)$, we see that $x - 1$ is negative and $\sqrt[3]{x+4}$ is also negative, and so $J'(x)$ is positive on this interval.
- On the interval $(-4, 0)$, we see that $x - 1$ is negative and $\sqrt[3]{x+4}$ is positive, and so $J'(x)$ is negative on this interval.
- On the interval $(0, 1)$, we see that $x - 1$ is negative and $\sqrt[3]{x+4}$ is positive, and so $J'(x)$ is negative on this interval.
- On the interval $(1, \infty)$, we see that $x - 1$ is positive and $\sqrt[3]{x+4}$ is also positive, and so $J'(x)$ is positive on this interval.

Since $J'(x)$ is positive to the left of $x = -4$ and negative to the right, we see $J'(x)$ has a $\boxed{\text{local maximum at } x = -4.}$ Since $J'(x)$ is negative to the left of $x = 0$ and negative to the right, we see that $J'(x)$ has neither a local maximum nor minimum at $x = 0$. Since $J'(x)$ is negative to the left of $x = 1$ and positive to the right, we see that $J'(x)$ has a $\boxed{\text{local minimum at } x = 1.}$

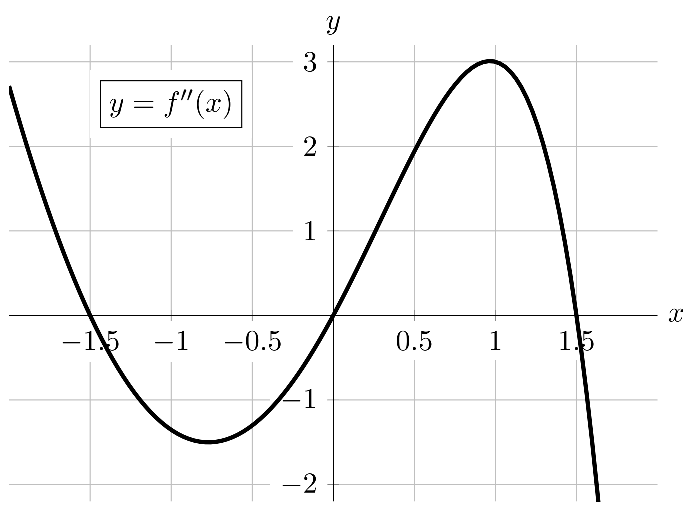
c. [4 points] The polynomial $8x^2 + 31x - 24$ (from the numerator of $J''(x)$) has two zeroes a and b, where $a \approx -4.54$ and $b \approx 0.66$. How many inflection points does the function $J(x)$ have? Remember to justify your answer. *Hint: What does the graph of* $8x^2 + 31x - 24$ *look like?*

Solution: The inflection points of $J(x)$ can only occur when $J''(x) = 0$ or $J''(x)$ does not exist. We see that $J''(x) = 0$ when $x = 0, a$, and b, and we see that $J''(x)$ does not exist when $x = -4$. We consider the signs of $J''(x)$ on the intervals between these points. Observe that $\left(\sqrt[3]{x+4}\right)^4$ is always positive unless $x = -4$. Also observe that the graph of $8x^2 + 31x - 24$ is a concave up parabola, which tells us that $8x^2 + 31x - 24$ is only negative when $a < x < b$.

- On the interval $(-\infty, a)$, we see that x is negative and $8x^2 + 31x - 24$ is positive, and so $J''(x)$ is negative.
- On the interval $(a, -4)$, we see that x is negative and $8x^2 + 31x - 24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(-4, 0)$, we see that x is negative and $8x^2 + 31x - 24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(0, b)$, we see that x is positive and $8x^2 + 31x - 24$ is negative, and so $J''(x)$ is negative.
- On the interval (b, ∞), we see that x is positive and $8x^2 + 31x - 24$ is also positive, and so $J''(x)$ is positive.

We observe that $J''(x)$ changes sign only at $x = a, 0$, and b, and so we conclude that $J''(x)$ has 3 inflection points.

9. [16 points] We consider a function $f(x)$ defined for all real numbers. We suppose that the first and second derivatives $f'(x)$ and $f''(x)$ are also defined for all real numbers. Below we show the graph of the **second derivative** of f. You may assume that $f''(x)$ is decreasing outside of the region shown.



a. [3 points] Find or estimate the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE.

Solution: The inflection points of $f(x)$ occur when $f''(x)$ changes sign. Therefore the x-coordinates of the inflection points of $f(x)$ are $x = -1.5, 0$, and 1.5.

b. [3 points] Find or estimate the x-coordinates of all inflection points of $f'(x)$. If there are none, write NONE.