8. [11 points] Suppose \( J(x) \) is a continuous function defined for all real numbers \( x \). The derivative and second derivative of \( J(x) \) are given by

\[
J'(x) = \frac{x^2(x - 1)}{\sqrt[3]{x + 4}} \quad \text{and} \quad J''(x) = \frac{x(8x^2 + 31x - 24)}{3(\sqrt[3]{x + 4})^4}.
\]

a. [2 points] Find the \( x \)-coordinates of all critical points of \( J(x) \). If there are none, write NONE.

Solution: The critical points of \( J(x) \) occur when \( J'(x) = 0 \) or \( J'(x) \) does not exist. We see that \( J'(x) = 0 \) when \( x = 0 \) and \( x = 1 \), and we see that \( J'(x) \) does not exist when \( x = -4 \). We conclude that the critical points of \( J(x) \) occur at \( x = 0, 1, \) and \( -4 \).

b. [5 points] Find the \( x \)-coordinates of
   i. all local minima of \( J(x) \) and
   ii. all local maxima of \( J(x) \).

If there are none of a particular type, write NONE.

Solution: We use the First Derivative Test. Observe that \( x^2 \) is always positive unless \( x = 0 \).
- On the interval \((-\infty, -4)\), we see that \( x - 1 \) is negative and \( \sqrt[3]{x + 4} \) is also negative, and so \( J'(x) \) is positive on this interval.
- On the interval \((-4, 0)\), we see that \( x - 1 \) is negative and \( \sqrt[3]{x + 4} \) is positive, and so \( J'(x) \) is negative on this interval.
- On the interval \((0, 1)\), we see that \( x - 1 \) is negative and \( \sqrt[3]{x + 4} \) is positive, and so \( J'(x) \) is negative on this interval.
- On the interval \((1, \infty)\), we see that \( x - 1 \) is positive and \( \sqrt[3]{x + 4} \) is also positive, and so \( J'(x) \) is positive on this interval.

Since \( J'(x) \) is positive to the left of \( x = -4 \) and negative to the right, we see \( J'(x) \) has a local maximum at \( x = -4 \). Since \( J'(x) \) is negative to the left of \( x = 0 \) and negative to the right, we see that \( J'(x) \) has neither a local maximum nor minimum at \( x = 0 \). Since \( J'(x) \) is negative to the left of \( x = 1 \) and positive to the right, we see that \( J'(x) \) has a local minimum at \( x = 1 \).
c. [4 points] The polynomial \(8x^2 + 31x - 24\) (from the numerator of \(J''(x)\)) has two zeroes \(a\) and \(b\), where \(a \approx -4.54\) and \(b \approx 0.66\). How many inflection points does the function \(J(x)\) have? Remember to justify your answer. **Hint:** What does the graph of \(8x^2 + 31x - 24\) look like?

**Solution:** The inflection points of \(J(x)\) can only occur when \(J''(x) = 0\) or \(J''(x)\) does not exist. We see that \(J''(x) = 0\) when \(x = 0, a, \) and \(b\), and we see that \(J''(x)\) does not exist when \(x = -4\). We consider the signs of \(J''(x)\) on the intervals between these points. Observe that \((\sqrt[3]{x+4})^4\) is always positive unless \(x = -4\). Also observe that the graph of \(8x^2 + 31x - 24\) is a concave up parabola, which tells us that \(8x^2 + 31x - 24\) is only negative when \(a < x < b\).

• On the interval \((-\infty, a)\), we see that \(x\) is negative and \(8x^2 + 31x - 24\) is positive, and so \(J''(x)\) is negative.

• On the interval \((a, -4)\), we see that \(x\) is negative and \(8x^2 + 31x - 24\) is also negative, and so \(J''(x)\) is positive.

• On the interval \((-4, 0)\), we see that \(x\) is negative and \(8x^2 + 31x - 24\) is also negative, and so \(J''(x)\) is positive.

• On the interval \((0, b)\), we see that \(x\) is positive and \(8x^2 + 31x - 24\) is negative, and so \(J''(x)\) is negative.

• On the interval \((b, \infty)\), we see that \(x\) is positive and \(8x^2 + 31x - 24\) is also positive, and so \(J''(x)\) is positive.

We observe that \(J''(x)\) changes sign only at \(x = a, 0,\) and \(b\), and so we conclude that \(J''(x)\) has 3 inflection points.

9. [16 points] We consider a function \(f(x)\) defined for all real numbers. We suppose that the first and second derivatives \(f'(x)\) and \(f''(x)\) are also defined for all real numbers. Below we show the graph of the **second derivative** of \(f\). You may assume that \(f''(x)\) is decreasing outside of the region shown.

![Graph of f''(x)](image)

a. [3 points] Find or estimate the \(x\)-coordinates of all inflection points of \(f(x)\). If there are none, write NONE.

**Solution:** The inflection points of \(f(x)\) occur when \(f''(x)\) changes sign. Therefore the \(x\)-coordinates of the inflection points of \(f(x)\) are \(x = -1.5, 0, \) and \(1.5\).

b. [3 points] Find or estimate the \(x\)-coordinates of all inflection points of \(f'(x)\). If there are none, write NONE.