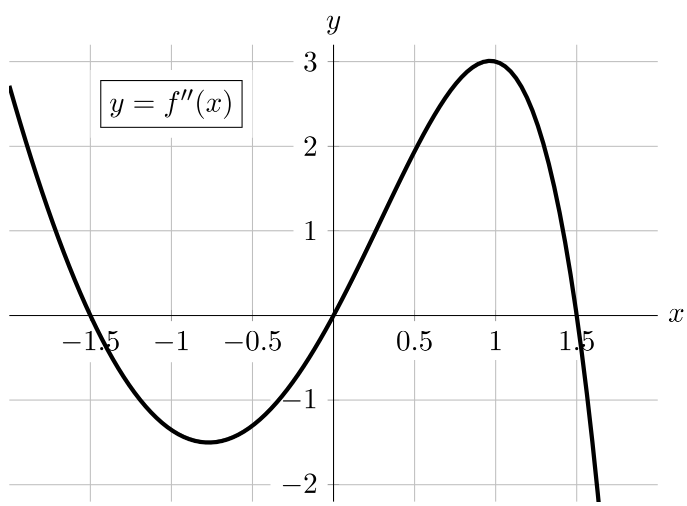
c. [4 points] The polynomial $8x^2 + 31x - 24$ (from the numerator of $J''(x)$) has two zeroes a and b, where $a \approx -4.54$ and $b \approx 0.66$. How many inflection points does the function $J(x)$ have? Remember to justify your answer. *Hint: What does the graph of* $8x^2 + 31x - 24$ *look like?*

Solution: The inflection points of $J(x)$ can only occur when $J''(x) = 0$ or $J''(x)$ does not exist. We see that $J''(x) = 0$ when $x = 0, a$, and b, and we see that $J''(x)$ does not exist when $x = -4$. We consider the signs of $J''(x)$ on the intervals between these points. Observe that $\left(\sqrt[3]{x+4}\right)^4$ is always positive unless $x = -4$. Also observe that the graph of $8x^2 + 31x - 24$ is a concave up parabola, which tells us that $8x^2 + 31x - 24$ is only negative when $a < x < b$.

- On the interval $(-\infty, a)$, we see that x is negative and $8x^2 + 31x - 24$ is positive, and so $J''(x)$ is negative.
- On the interval $(a, -4)$, we see that x is negative and $8x^2 + 31x - 24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(-4, 0)$, we see that x is negative and $8x^2 + 31x - 24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(0, b)$, we see that x is positive and $8x^2 + 31x - 24$ is negative, and so $J''(x)$ is negative.
- On the interval (b, ∞), we see that x is positive and $8x^2 + 31x - 24$ is also positive, and so $J''(x)$ is positive.

We observe that $J''(x)$ changes sign only at $x = a, 0$, and b, and so we conclude that $J''(x)$ has **3 inflection points.**

9. [16 points] We consider a function $f(x)$ defined for all real numbers. We suppose that the first and second derivatives $f'(x)$ and $f''(x)$ are also defined for all real numbers. Below we show the graph of the **second derivative** of f. You may assume that $f''(x)$ is decreasing outside of the region shown.



a. [3 points] Find or estimate the x-coordinates of all inflection points of $f(x)$. If there are none, write NONE.

Solution: The inflection points of $f(x)$ occur when $f''(x)$ changes sign. Therefore the x-coordinates of the inflection points of $f(x)$ are $x = -1.5, 0$, and 1.5.

b. [3 points] Find or estimate the x-coordinates of all inflection points of $f'(x)$. If there are none, write NONE.

Solution: The inflection points of $f'(x)$ occur when $f'''(x)$ changes sign. Since $f'''(x)$ is the first derivative of $f''(x)$, this means that we are looking for points where the graph changes from increasing to decreasing, or vice versa. These occur at $\boxed{x \approx -0.75 \text{ and } x \approx 0.9.}$

c. [1 point] Suppose that $f'(0) = 5$. How many critical points does f have?

Solution: By looking at the graph, we see that $f'(x)$ is increasing on the intervals $(-\infty, -1.5)$ and $(0, 1.5)$, and is decreasing on the intervals $(-1.5, 0)$ and $(1.5, 0)$. This tells us that $f'(x)$ has a local minimum at $x = 0$ and local maxima at $x = -1.5$ and $x = 1.5$. Since $f'(0) = 5$, this means that $f'(x)$ cannot be equal to 0 on the interval $(-1.5, 1.5)$, because it is always positive on this interval.

Since $f''(x)$ is decreasing and positive on the interval $(-\infty, -1.5)$, we see that $f'(x)$ is increasing and concave down on this interval. Therefore $f'(x)$ diverges to $-\infty$ as $x \to -\infty$. Since $f'(-1.5)$ is positive, we conclude that $f'(x)$ must cross the x-axis *once* on the interval $(-\infty, -1.5)$. Similarly, since $f''(x)$ is decreasing and negative on the interval $(1.5, \infty)$, we see that $f'(x)$ is decreasing and concave down on this interval. Therefore $f'(x)$ diverges to $-\infty$ as $x \to \infty$. Since $f'(1.5)$ is positive, we conclude that $f'(x)$ must cross the x-axis *once* on the interval $(1.5, \infty)$.

We conclude that $f'(x)$ is equal to 0 at exactly two x-values, and so the function f has $\boxed{\text{2 critical points.}}$

For parts **d.-f.** below, suppose that $f'(1) = 6.8$ and $f(1) = 4$.

d. [4 points] Let $Q(x)$ be the quadratic approximation of $f(x)$ near $x = 1$. Find a formula for $Q(x)$.

Solution: We have

$$\begin{aligned} Q(x) &= \frac{f''(1)}{2}(x-1)^2 + f'(1)(x-1) + f(1) \\ &= \boxed{\frac{3}{2}(x-1)^2 + 6.8(x-1) + 4.} \end{aligned}$$

e. [2 points] Is the linear approximation of $f(x)$ near $x = 1$ an overestimate or an underestimate of $f(x)$ for values of x near 1? Explain your reasoning.

Solution: Since $f''(1) > 0$, it follows that the linear approximation of $f(x)$ near $x = 1$ is an $\boxed{\text{underestimate}}$ of $f(x)$ for values of x near 1.

f. [3 points] Let $L(x)$ be the linear approximation of $f'(x)$ (the derivative of f) near $x = 1$. Find a formula for $L(x)$.

Solution: We have

$$\begin{aligned} L(x) &= f''(1)(x-1) + f'(1) \\ &= \boxed{3(x-1) + 6.8.} \end{aligned}$$