c. [4 points] The polynomial $8x^2 + 31x - 24$ (from the numerator of $J''(x)$) has two zeroes $a$ and $b$, where $a \approx -4.54$ and $b \approx 0.66$. How many inflection points does the function $J(x)$ have? Remember to justify your answer. *Hint: What does the graph of $8x^2 + 31x - 24$ look like?*

**Solution:** The inflection points of $J(x)$ can only occur when $J''(x) = 0$ or $J''(x)$ does not exist. We see that $J''(x) = 0$ when $x = 0, a,$ and $b$, and we see that $J''(x)$ does not exist when $x = -4$. We consider the signs of $J''(x)$ on the intervals between these points. Observe that $(\sqrt{x} + 4)^4$ is always positive unless $x = -4$. Also observe that the graph of $8x^2 + 31x - 24$ is a concave up parabola, which tells us that $8x^2 + 31x - 24$ is only negative when $a < x < b$.

- On the interval $(-\infty, a)$, we see that $x$ is negative and $8x^2 + 31x - 24$ is positive, and so $J''(x)$ is negative.
- On the interval $(a, -4)$, we see that $x$ is negative and $8x^2 + 31x - 24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(-4, 0)$, we see that $x$ is negative and $8x^2 + 31x - 24$ is also negative, and so $J''(x)$ is positive.
- On the interval $(0, b)$, we see that $x$ is positive and $8x^2 + 31x - 24$ is negative, and so $J''(x)$ is negative.
- On the interval $(b, \infty)$, we see that $x$ is positive and $8x^2 + 31x - 24$ is also positive, and so $J''(x)$ is positive.

We observe that $J''(x)$ changes sign only at $x = a, 0,$ and $b$, and so we conclude that $J''(x)$ has 3 inflection points.

9. [16 points] We consider a function $f(x)$ defined for all real numbers. We suppose that the first and second derivatives $f'(x)$ and $f''(x)$ are also defined for all real numbers. Below we show the graph of the second derivative of $f$. You may assume that $f''(x)$ is decreasing outside of the region shown.

![Graph of f''(x)](image)

a. [3 points] Find or estimate the $x$-coordinates of all inflection points of $f(x)$. If there are none, write none.

**Solution:** The inflection points of $f(x)$ occur when $f''(x)$ changes sign. Therefore the $x$-coordinates of the inflection points of $f(x)$ are $x = -1.5, 0,$ and $1.5$.

b. [3 points] Find or estimate the $x$-coordinates of all inflection points of $f'(x)$. If there are none, write none.
Solution: The inflection points of \( f'(x) \) occur when \( f''''(x) \) changes sign. Since \( f''''(x) \) is the first derivative of \( f''(x) \), this means that we are looking for points where the graph changes from increasing to decreasing, or vice versa. These occur at \( x \approx -0.75 \) and \( x \approx 0.9 \).

c. [1 point] Suppose that \( f'(0) = 5 \). How many critical points does \( f \) have?

\[ \text{Solution: By looking at the graph, we see that } f'(x) \text{ is increasing on the intervals } (-\infty, -1.5) \text{ and } (0, 1.5), \text{ and is decreasing on the intervals } (-1.5, 0) \text{ and } (1.5, 0). \text{ This tells us that } f'(x) \text{ has a local minimum at } x = 0 \text{ and local maxima at } x = -1.5 \text{ and } x = 1.5. \text{ Since } f'(0) = 5, \text{ this means that } f'(x) \text{ cannot be equal to 0 on the interval } (-1.5, 1.5), \text{ because it is always positive on this interval.}\]

We conclude that \( f'(x) \) is equal to 0 at exactly two \( x \)-values, and so the function \( f \) has 2 critical points.

For parts d.-f. below, suppose that \( f'(1) = 6.8 \) and \( f(1) = 4 \).

d. [4 points] Let \( Q(x) \) be the quadratic approximation of \( f(x) \) near \( x = 1 \). Find a formula for \( Q(x) \).

\[ \text{Solution: We have} \]
\[ Q(x) = \frac{f''''(1)}{2} (x - 1)^2 + f'(1)(x - 1) + f(1) \]
\[ = \frac{3}{2}(x - 1)^2 + 6.8(x - 1) + 4. \]

e. [2 points] Is the linear approximation of \( f(x) \) near \( x = 1 \) an overestimate or an underestimate of \( f(x) \) for values of \( x \) near 1? Explain your reasoning.

\[ \text{Solution: Since } f''''(1) > 0, \text{ it follows that the linear approximation of } f(x) \text{ near } x = 1 \text{ is an underestimate of } f(x) \text{ for values of } x \text{ near 1.} \]

f. [3 points] Let \( L(x) \) be the linear approximation of \( f'(x) \) (the \text{derivative of } f) \text{ near } x = 1. \text{ Find a formula for } L(x).

\[ \text{Solution: We have} \]
\[ L(x) = f''''(1)(x - 1) + f'(1) \]
\[ = 3(x - 1) + 6.8. \]