6. [9 points] The Loads-of-Oats company is designing a new cylindrical container for their steel-cut oats. The company specifies that

- the height of the cylinder and four times the radius of the cylinder should sum to 18 inches
- the radius of the cylinder will be at least 1 inch, and
- the height of the cylinder will be at least 2 inches.

a. [2 points] What is the largest possible radius of such a cylindrical container?

Solution: Let r be the radius of the cylindrical container and h be the height of the cylindrical container. We require $h + 4r = 18$ and $h \geq 2$. Therefore, $18 - 4r \geq 2$ so $16 \geq 4r$ and $4 \geq r$. and so the largest possible radius of such a cylindrical container is 4 inches.

Answer: **4 inches**

b. [7 points] Find the height and radius of such a cylindrical container, in inches, that maximize the volume of the container.

In your solution, make sure to carefully define any variables and functions you use. Use calculus to justify your answers, and show enough evidence that the values you find do in fact maximize the volume.

Solution: As in part a. let r be the radius of the cylindrical container and h be the height of the cylindrical container. The volume of such a cylinder is $\pi r^2 h$. Using the constraint that $h = 18 - 4r$, the volume $V(r)$ of the cylindrical container is given by

$$V(r) = \pi r^2(18 - 4r) = 18\pi r^2 - 4\pi r^3.$$

Because $r \geq 1$ and $r \leq 4$, the domain of $V(r)$ is $[1, 4]$. The derivative of $V(r)$ is

$$V'(r) = 36\pi r - 12\pi r^2.$$

The critical points of $V(r)$ occur when $V'(r) = 0$ or $V'(r)$ does not exist. The latter does not occur so checking when $V'(r) = 0$, we find

$$\begin{aligned} 36\pi r - 12\pi r^2 &= 0 \\ 12\pi r(3 - r) &= 0 \\ \text{So } r = 0 &\text{ or } r = 3. \end{aligned}$$

However, $r = 0$ is not in the domain of the function $V(r)$ because $r = 0$ is not in $[1, 4]$. Therefore, the only critical point of $V(r)$ in this context is $r = 3$. Because $V(r)$ is a continuous function on the closed interval $[1, 4]$, the Extreme Value Theorem (EVT) guarantees a global maximum. So, to find the global maximum, we compare the values of $V(r)$ at the endpoints and critical point.

r	$V(r) = \pi r^2(18 - 4r)$
1	$\pi(18 - 4) = 12\pi$
3	$9\pi(18 - 12) = 54\pi$
4	$16\pi(18 - 16) = 32\pi$

Therefore, $V(r)$ attains its global maximum at $r = 3$. So a radius $r = 3$ inches and a height of $h = 18 - 4(3) = 6$ inches maximizes the volume of the container, and the resulting maximum volume is 54π cubic inches.

Answer: height = **6 inches** and radius = **3 inches**