8. [8 points] Prairie dogs named Paws and Dot have been hard at work digging a tunnel. Consider the functions L and C defined as follows:

- $L(w)$ is the length of the tunnel, in feet, when w pounds of dirt have been removed.
- $C(w)$ is the total number of Calories the prairie dogs have burned digging their tunnel when they have removed a total of w pounds of dirt for their tunnel.

The functions $L(w)$ and $C(w)$ are both invertible and differentiable.

a. [4 points] Complete the sentence below to give a practical interpretation of the equation

$$(L^{-1})'(10) = 24.$$

In order to increase the length of the tunnel from 10 *feet to* 10.25 *feet, ...*

Solution: the prairie dogs have to remove approximately 6 pounds of dirt.

b. [4 points]

i. Which of the following expressions gives the length, in feet, of the prairie dog tunnel when the prairie dogs have burned a total of x Calories digging? Circle the one correct expression.

$C(L^{-1}(x))$ $C^{-1}(L(x))$ $\boxed{L(C^{-1}(x))}$ $L^{-1}(C(x))$

ii. Use the answer you selected in part i to find an expression for the instantaneous rate of change of the length of the prairie dog tunnel, in feet per calorie, when the prairie dogs have burned a total of 2000 calories digging.
Simplify as much as possible. Note that your final answer may involve the function names L, L^{-1}, L' , C, C^{-1}, *and* C' *but should not involve the function names* $\left(L^{-1}\right)'$ *or* $\left(C^{-1}\right)'$.

Solution: We are trying to find a simplified expression for $\frac{d}{dx}\left(L\left(C^{-1}(x)\right)\right)$ at $x = 2000$. Using the chain rule and the formula for the derivative of the inverse of a function, we have

$$\begin{aligned}\frac{d}{dx}\left(L\left(C^{-1}(x)\right)\right) &= L'\left(C^{-1}(x)\right)\cdot\left(C^{-1}\right)'(x)\\ &= L'\left(C^{-1}(x)\right)\cdot\frac{1}{C'\left(C^{-1}(x)\right)}.\end{aligned}$$

So at $x = 2000$, we have

$$L'\left(C^{-1}(2000)\right)\cdot\frac{1}{C'\left(C^{-1}(2000)\right)} = \frac{L'\left(C^{-1}(2000)\right)}{C'\left(C^{-1}(2000)\right)}.$$

Answer: $L'\left(C^{-1}(2000)\right)\cdot\frac{1}{C'\left(C^{-1}(2000)\right)} = \frac{L'\left(C^{-1}(2000)\right)}{C'\left(C^{-1}(2000)\right)}$