7. [8 points] Allie, Minnie, and Ian have decided to start a business producing soda cans. They've received their first order of 100,000 cans, but haven't settled on a final product design yet. They want their cylindrical cans to hold 355 mL of liquid, so if the cans have base radius r and height h, both in centimeters, then by the formula for the volume of a cylinder,

$$\pi r^2 h = 355.$$

Given the cost of aluminum, Allie, Minnie, and Ian calculate that producing 100,000 cans with a base radius r and height h would cost

$$k\pi r(r+h)$$

dollars, where $k = 12.96$.

a. [2 points] For a given radius r, let $C(r)$ equal the cost of producing 100,000 aluminum cans with radius r and a volume of 355 mL. Find a formula for $C(r)$. Your formula should not include h, but can include k.

Answer: $C(r) =$ $k\pi r\left(r + \dfrac{355}{\pi r^2}\right) = k\pi r^2 + \dfrac{355k}{r}$

b. [1 point] What is the domain of $C(r)$?

Answer: $(0, \infty)$

c. [5 points] Find the radius r that minimizes the cost of producing 100,000 soda cans. Use calculus to find and justify your answer, and be sure to show enough evidence that the value(s) you find do in fact minimize the cost.

Solution: Differentiating $C(r)$ gives us $C'(r) = 2k\pi r - 355k/r^2$, which is defined everywhere on $(0, \infty)$ and equals 0 exactly when

$$2k\pi r = \frac{355k}{r^2}, \quad \text{that is, when} \quad r = \sqrt[3]{\frac{355}{2\pi}}.$$

This is the only critical point of $C(r)$ on $(0, \infty)$. Since the domain of $C(r)$ does not include any endpoints, if $C(r)$ has a minimum (or a maximum) then it must be at this point. Checking the one-sided limits of $C(r)$ at the endpoints of its domain gives

$$\lim_{r\to 0^+} C(r) = \infty = \lim_{r\to\infty} C(r),$$

so the critical point $\sqrt[3]{\frac{355}{2\pi}}$ must be the global minimum of $C(r)$ on $(0, \infty)$, and is therefore the radius that minimizes cost.

Answer: value(s) of r that minimize the cost: $\sqrt[3]{\dfrac{355}{2\pi}}$