8. [6 points] The equation $x^2 + xy + 2y^2 = 28$ defines y implicitly as a function of x.

a. [4 points] Compute $\frac{dy}{dx}$. Show every step of your work.

Solution: Implicitly differentiating the equation $x^2 + xy + 2y^2 = 28$ with respect to x gives:

$$2x + y + x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0.$$

Now collect like terms, and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(x + 4y\right) = -2x - y, \qquad \text{so} \qquad \frac{dy}{dx} = \frac{-2x - y}{x + 4y}.$$

Answer: $\frac{dy}{dx} = \frac{-2x - y}{x + 4y}$

b. [2 points] Find an equation of the line tangent to the curve defined by $x^2 + xy + 2y^2 = 28$ at the point $(2, 3)$.

Solution: Using point-slope form, the tangent line will have equation $y - 3 = m(x - 2)$, where m is the slope. And, using part (a), the slope of the tangent line at $(2, 3)$ is

$$m = \frac{dy}{dx}\Big|_{(x,y)=(2,3)} = \frac{-2(2) - 3}{2 + 4(3)} = \frac{-7}{14} = -\frac{1}{2}.$$

So the line has equation $y - 3 = -\frac{1}{2}(x - 2)$.

Answer: $y - 3 = -\frac{1}{2}(x - 2)$

9. [6 points] The equation $x + \frac{1}{3}y^3 - y = 1$ implicitly defines x and y as functions of each other. Implicitly differentiating this equation with respect to x and solving for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx} = \frac{-1}{y^2 - 1}.$$

Let $\mathcal{C}$ be the graph of the equation $x + \frac{1}{3}y^3 - y = 1$. *Note that all points listed as possible answers below do actually lie on the graph* $\mathcal{C}$.

a. [2 points] Circle all points below at which the line tangent to $\mathcal{C}$ is *horizontal.*

$(-5, 3)$ $\left(\frac{1}{3}, -1\right)$ $(1, 0)$ $\left(1 + \frac{\sqrt{2}}{3}, \sqrt{2}\right)$ $\left(\frac{5}{3}, 1\right)$ $\boxed{\text{NONE OF THESE}}$

b. [2 points] Circle all points below at which the line tangent to $\mathcal{C}$ is *vertical.*

$(-5, 3)$ $\boxed{\left(\frac{1}{3}, -1\right)}$ $(1, 0)$ $\left(1 + \frac{\sqrt{2}}{3}, \sqrt{2}\right)$ $\boxed{\left(\frac{5}{3}, 1\right)}$ NONE OF THESE

c. [2 points] Circle all points below at which $\frac{dy}{dx}$ and $\frac{dx}{dy}$ are equal to each other.

$(-5, 3)$ $\left(\frac{1}{3}, -1\right)$ $\boxed{(1, 0)}$ $\boxed{\left(1 + \frac{\sqrt{2}}{3}, \sqrt{2}\right)}$ $\left(\frac{5}{3}, 1\right)$ NONE OF THESE

Solution: $\frac{dy}{dx} = \frac{dx}{dy}$ when $\frac{-1}{y^2-1} = \frac{y^2-1}{-1}$, which happens when $(y^2 - 1)^2 = 1$. Thus $y^2 = 2$ or $y^2 = 0$, which means $y = 0$ or $y = \pm\sqrt{2}$.