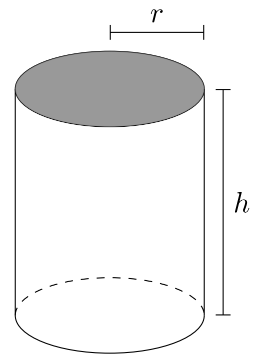
4. [8 points]



Alana is designing a new prototype for her Stan Lee cups. The new cups are cylindrical in shape, with metal sides and base, and a circular lid made from silicone. If the cylinder has height h centimeters, and radius r centimeters, then the surface area of the metal part is $2\pi rh + \pi r^2$ square centimeters, and the surface area of the silicone part is πr^2. The metal costs 2 cents per square centimeter, and the silicone costs 3 cents per square centimeter. Alana spends a total of 300 cents on materials for each cup.

a. [3 points] Find a formula for h in terms of r.

Solution: The total cost in cents of materials for each cup is

$$300 = 2\left(2\pi rh + \pi r^2\right) + 3\pi r^2 = 4\pi rh + 5\pi r^2.$$

Therefore, $4\pi rh = 300 - 5\pi r^2$, so $h = \dfrac{300 - 5\pi r^2}{4\pi r} = \dfrac{75}{\pi r} - \dfrac{5r}{4}$.

Answer: $h = \dfrac{300 - 5\pi r^2}{4\pi r}$

b. [1 point] Recall that the volume of a cylinder of radius r and height h is $V = \pi r^2 h$. Write a formula for $V(r)$, the volume of one of the cups in cubic centimeters, as a function of r only. *Your formula should not include the letter h.*

Answer: $V(r) = \pi r^2 \cdot \dfrac{300 - 5\pi r^2}{4\pi r} = r \cdot \dfrac{300 - 5\pi r^2}{4}$

c. [4 points] Alana wants to ensure that the height of a cup is at most 2 and a half times its radius, that is, she wants $h \leq 2.5r$. Given this constraint, find the domain of $V(r)$ in the context of this problem.

Solution: In the context of this problem, the height should be positive, so we have the constraints $0 < h \leq 2.5r$. Substituting in our expression for h from part **a**., we get:

$$\begin{aligned} 0 &< \frac{300 - 5\pi r^2}{4\pi r} \leq 2.5r \\ 0 &< 300 - 5\pi r^2 \leq 10\pi r^2 \\ 5\pi r^2 &< 300 \leq 15\pi r^2. \end{aligned}$$

Therefore, we must have

$$\frac{300}{15\pi} \leq r^2 < \frac{300}{5\pi}, \qquad \text{which means} \qquad \sqrt{\frac{20}{\pi}} \leq r < \sqrt{\frac{60}{\pi}}.$$

Answer: $\left[\sqrt{\dfrac{20}{\pi}}, \sqrt{\dfrac{60}{\pi}}\right)$