7. [12 points] Let $f(x) = x^4e^x + 4e$ and $g(x) = -x^2 + (2+5e)x - 1$, and let $h(x)$ be the piecewise function

$$h(x) = \begin{cases} f(x) & x \leq 1 \\ g(x) & x > 1. \end{cases}$$

Note that $f(1) = 5e = g(1)$ and $f'(1) = 5e = g'(1)$, so $h(x)$ is continuous and differentiable at $x = 1$. To answer the questions below, you may use the following:

$$f'(x) = x^3e^x(x+4) \qquad \text{and} \qquad f''(x) = x^2e^x(x+2)(x+6).$$

a. [3 points] Find all critical points of $h(x)$. *No justification necessary.*

Solution: We are given that $h'(1)$ exists and is nonzero, so $x = 1$ is not a critical point of $h(x)$. This means the critical points of $h(x)$ are the critical points of $f(x)$ in $(-\infty, 1)$ together with the critical points of $g(x)$ in $(1, \infty)$. This includes $x = -4$, $x = 0$, and also $x = 1 + \frac{5e}{2}$ since $1 + \frac{5e}{2} > 1$.

Answer: $h(x)$ has critical points at $x =$ $-4, \ 0, \ 1 + \frac{5e}{2}$

b. [3 points] Find all critical points of $h'(x)$. *No justification necessary.*

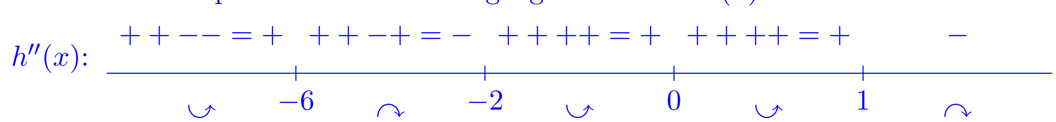
Solution: Since $g'(x)$ has no critical points on $(1, \infty)$, the critical points of $h'(x)$ will be the critical points of $f'(x)$ on $(-\infty, 1)$, together with $x = 1$ if $h''(1)$ is zero or undefined. Since $f''(1) = 21e \neq -2 = g''(1)$, we see that $h''(1)$ does not exist, so $x = 1$ is a critical point of $h'(x)$. The other critical points of $h'(x)$ are the critical points of $f'(x)$ on $(-\infty, 1)$, which are $x = -6$, $x = -2$, and $x = 0$.

Answer: $h'(x)$ has critical points at $x =$ $-6, \ -2, \ 0, \ 1$

c. [6 points] Find all inflection points of $h(x)$. *Show all your work. Be sure you show enough evidence to justify your conclusions.*

Solution: The inflection points of $h(x)$ occur at x-values where the graph of $h(x)$ changes concavity, that is, where $h''(x)$ changes sign from positive to negative or vice versa. So we need to determine the sign of $h''(x)$ on the five intervals determined by the four critical points of $h'(x)$ that we found in part **b**.

For $x < 1$, we have $h''(x) = f''(x)$, where $f''(x)$ is a product of the four terms x^2, e^x, $(x+2)$, and $(x+6)$, the first two of which are positive in each interval. For $x > 1$, we have $h''(x) = g''(x) = -2 < 0$. This produces the following sign chart for $h''(x)$:



The graph of $h(x)$ is concave up on intervals where $h''(x) > 0$, and concave down on intervals where $h''(x) < 0$, so the inflection points of $h(x)$ are at $x = -6$, $x = -2$, and $x = 1$, where $h''(x)$ changes sign. The critical point $x = 0$ of $h'(x)$ is not an inflection point of $h(x)$, since $h(x)$ is concave up on both $(-2, 0)$ and $(0, 1)$.

Answer: $h(x)$ has inflection points at $x =$ $-6, \ -2, \ 1$